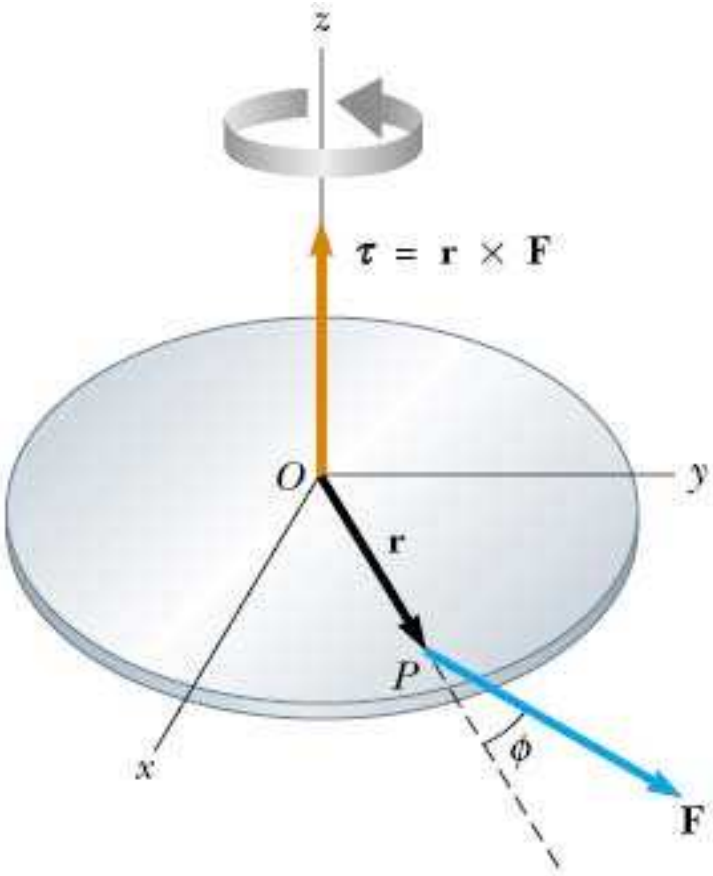
AL_IMAM MUHAMMAD BIN SAUD ISLAMIC UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF PHYSICS

CLASSICAL MECHANICS



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Chapter 9

Linear Momentum and Collisions

Consider what happens when a bowling ball strikes a pin. The pin is given a large velocity as a result of the collision; consequently, it flies away and hits another pins or is projected toward the backstop. Because the average force exerted on the pin during the collision is large, the pin achieves the large velocity very rapidly and experiences the force for a very short time interval. According to Newton's third law, the pin exerts a reaction force on the ball that is equal in magnitude and opposite in direction to the force exerted by the ball on the pin.
First, we introduce the concept of *momentum*, which is useful for describing objects in motion. The momentum of an object is related to both its mass and velocity. The concept of momentum leads us to a second conservation law, that of conservation of momentum. In this chapter we also introduce the concept of centre of mass of a system of particles.

9.1 Linear Momentum and Its Conservation

Let us apply the General Problem-Solving Strategy and *conceptualize* an isolated system of two particles (Fig. 9.1) with masses m_1 and m_2 and moving with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ at instant of time. Because the system is isolated, the only force on one particle is that from the other particle and we can *categorize* this as a situation in which Newton's laws will be useful. From Newton's third law of action-reaction pair, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. We can express this condition as

$$\mathbf{F}_{12} + \mathbf{F}_{21} = 0$$

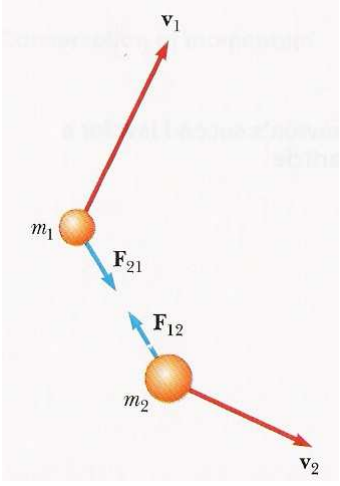


Figure 9.1 Two particles interact with each other. According to Newton's third law, we must have $\mathbf{F}_{12} = -\mathbf{F}_{21}$.

Let us further *analyze* this situation by incorporating Newton's second law, replacing each force with $m\mathbf{a}$ gives

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 = 0$$

Now we replace the acceleration with its definition from the Equation below

$$a \equiv \lim_{\Delta t \to 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}$$

We get:

$$m_1 \frac{dV_1}{dt} + m_2 \frac{dV_2}{dt} = 0$$

If the masses m_1 and m_2 are constant, we can bring them into the derivatives, which gives

$$\frac{d(m_1 V_1)}{dt} + \frac{d(m_2 V_2)}{dt} = 0$$

$$\frac{d}{dt}(m_1 V_1 + m_2 V_2) = 0 \qquad (9.1)$$

Note that the derivative of the sum $m_1 V_1 + m_2 V_2$ with respect to time is zero. Consequently, this sum must be constant. Therefore, the sum of these quantities for an isolated system is conserved. We call this quantity *linear momentum:*

The **linear momentum** of a particle or an object that can be modeled as a particle of mass m moving with the velocity $\boldsymbol{V}$ is defined to be the product of the mass and velocity:

$$\mathbf{p} = m\boldsymbol{V} \qquad (9.2)$$

Linear momentum is a vector quantity because it equals the product of a scalar quantity m and a vector quantity $\boldsymbol{V}$. Its direction is a long $\boldsymbol{V}$, it has dimensions ML/T, and its SI unit is kg.m/s.
If a particle is moving in an arbitrary direction, **p** must have three components, and Equation 9.2 is equivalent to the component equations

$$P_x = mv_x \qquad P_y = mv_y \qquad P_z = mv_z$$

As you can see from its definition, the concept of momentum provides a quantitative distinction between heavy and light particles moving at the same velocity. For example, the momentum of the bowling ball moving at 10 m/s is much grater than that of tennis ball moving at the same speed.
We start from Newton's second law and substitute the definition of acceleration:

$$\sum F = ma = m\frac{dV}{dt}$$

In Newton's second law, the mass m is assumed to be constant. Thus, we can bring m inside the derivative notation to give us

$$\sum F = \frac{d(mV)}{dt} = \frac{dP}{dt} \qquad (9.3)$$

This shows that **the time rate of change of the linear momentum of a particle is equal to the net force acting on the particle.**
This is alternative form of Newton's second law. We can use Equation 9.3 to study the phenomena in which the mass changes. For example, the mass of the rocket changes

as fuel is burned and ejected from the rocket. Equation 9.3 leads to a law of conservation of momentum for an isolated system.
Using the definition of momentum, Equation 9.1 can be written

$$\frac{d}{dt}(p_1 + p_2) = 0$$

Because the time derivative of the total momentum $p_{tot} = p_1 + p_2$ is *zero*, we conclude that the *total* momentum of the system must remain constant:

$$P_{tot} = P_1 + P_2 = constant \tag{9.4}$$

or, equivalently,

$$P_{1i} + P_{2i} = P_{1f} + P_{2f} \tag{9.5}$$

Where P_{1i} and P_{2i} are the initial values and P_{1f} and P_{2f} the final values of the momenta. Equation 9.5 in component form demonstrates that the total momenta in the x, y, and z direction are all independently conserved:

$$P_{ix} = P_{fx} \quad P_{iy} = P_{fy} \quad P_{iz} = P_{fz} \tag{9.6}$$

This result, known as the **law of conservation of linear momentum**: **whenever two or more particles in an isolated system interact, the total momentum of the system remains constant.**

9.2 Impulse and Momentum

If a single force **F** acts on a particle and that this force may vary with time, according to Newton's second law, $F = dp / dt$, or

$$dp = Fdt \tag{9.7}$$

If the momentum of the particle changes from $\mathbf{p}_i$ at time t_i to $\mathbf{p}_f$ at time t_f, integrating Equation 9.7 gives

$$\Delta p = p_f - p_i = \int_{t_i}^{t_f} F dt \tag{9.8}$$

The quantity on the right side of this equation is called the **impulse**. Impulse is a vector defined by

$$I \equiv \int_{t_i}^{t_f} F dt \qquad \textbf{(Impulse of force)} \tag{9.9}$$

Equation 9.8 is known as the **impulse-momentum theorem: The impulse of the force F acting on a particle equals the change in the momentum of the particle.**

This statement is equivalent to Newton's second law. From this definition, we see that impulse is a vector quantity having a magnitude equal to the area under the force-time curve, as described in Figure 9.2a.

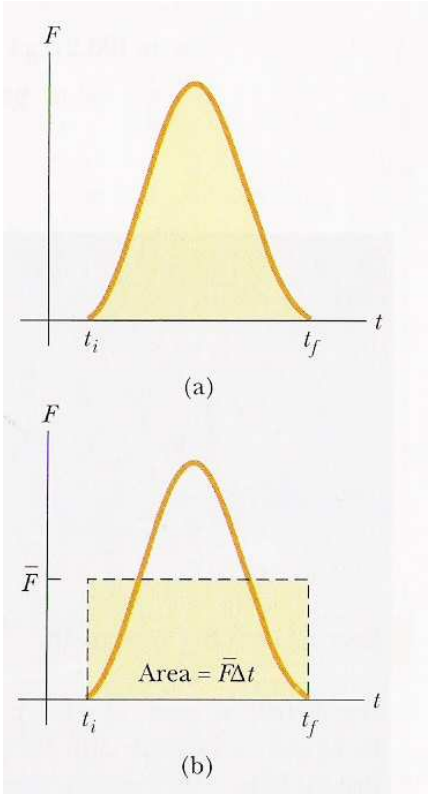


Figure 9.2 (a) A force acting on a particle may vary in time. The impulse imparted to the particle by the force is the area under the force-versus-time curve. (b) in the time interval Δt, the time averaged force (horizontal dashed line) gives the same impulse to a particle as does the time-varying force described in part (a).

Impulse has the dimensions of momentum—that is, ML/T.
Because the force imparting an impulse can generally vary in time, it is convenient to define a time-averaged force

$$\overline{F} \equiv \frac{1}{\Delta t}\int_{t_i}^{t_f} F dt \tag{9.10}$$

Therefore, we can express Equation 9.9 as

$$I = \overline{F}\Delta t \tag{9.11}$$

The calculation of impulse becomes especially simple if the force acting on the particle is constant. In this case, $\overline{F} = F$ and Equation 9.11 becomes

$$I = F\Delta t \tag{9.12}$$

In many physical situations, we shall use what is called the **impulse approximation, in which we assume that one of the forces exerted on a particle acts for a short time but is much greater than any other force present.** This approximation is especially useful in treating collisions in which the duration of the collision is very short. When this approximation is made, we refer to the force as an *impulse force*. For example, when a baseball is struck with a bat, the time of the collision is about 0.01 s and the average force that the bat exerts on the ball in this time is typically several thousand newtons. When we use this approximation it is important to remember that $\mathbf{p}_i$ and $\mathbf{p}_f$ represent the momenta *immediately* before and after the collision, and the particle moves very little during the collision.

9.3 Collisions in One Dimension

We use the term **collision** to represent an event during which two particles come close to each other and interact by means of forces. The time interval during which the velocities of the particles change from initial to final values is assumed to be short.
A collision may involve a physical contact as in Figure 9.3a, but the "physical contact" on a submicroscopic scale is ill-defined and hence meaningless. To understand this, consider a collision on atomic scale (Figure 9.3b), such as the collision of the proton with alpha particle (the nucleus of a helium atom). Because the particles are both positively charged, they never come in to "physical contact". The total momentum of an isolated system just before a collision equals the total momentum of the system just after the collision.

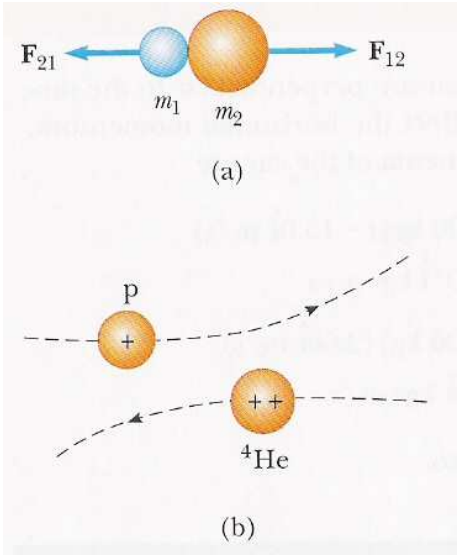


Figure 9.3 (a) The collision between two object s as the result of direct contact. (b) The "collision" between two charges particles.

An **elastic collision** between two objects is one in which **the total kinetic energy (as well as total momentum) of the system is the same before and after collision.**

An **inelastic collision** is one in which **the total kinetic energy of the system is not the same before and after the collision (even though the momentum of the system is conserved).**
Inelastic collisions are of two types:

1. When the colliding objects stick together after the collision, the collision is called **perfectly inelastic.**
2. When the colliding objects do not stick together, but some of the kinetic energy is lost, the collision is called **inelastic**.

In most collision, the kinetic energy of the system is *not* conserved because some of the energy is converted to internal energy and some of it is transferred away by mean of sound. Elastic and perfectly inelastic collisions are limiting case, most collisions fall some where between.
The important distinguish between these two types of collisions is that **momentum of the system is conserved in all collisions, but kinetic energy of the system is conserved only in elastic collisions.**

Perfectly Inelastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities V_{1i} and V_{2i} along the same straight line, as in Figure 9.4.

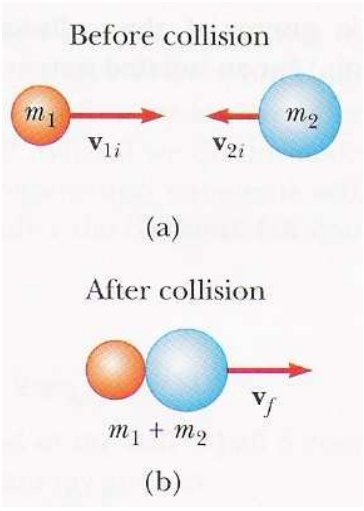


Figure 9.4 Schematic representation of a perfectly inelastic head-on collision between two particles: (a) before collision and (b) after collision.

The particles collide head-on, stick together, and then move with some common velocity $\boldsymbol{V}_f$ after the collision. Because the momentum of an isolated system is conserved in *any* collision, we can say that the total momentum before collision equals the total momentum of the composite system after collision:

$$m_1 V_{1i} + m_2 V_{2i} = (m_1 + m_2) V_f \tag{9.13}$$

Solving for the final velocity gives

$$V_f = \frac{m_1 V_{1i} + m_2 V_{2i}}{m_1 + m_2} \tag{9.14}$$

Elastic Collisions

Consider two particles of masses m_1 and m_2 moving with initial velocities V_{1i} and V_{2i} along the same straight line, as in Figure 9.5

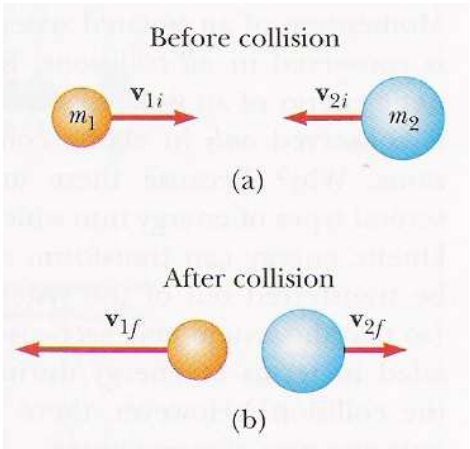


Figure 9.5 Schematic representation of a perfectly elastic head-on collision between two particles: (a) before collision and (b) after collision.

The two particles collide head-on, and then leave the collision site with different velocities, $\boldsymbol{V}_{1f}$ and $\boldsymbol{V}_{2f}$. If the collision is elastic, both the momentum and kinetic energy of the system are conserved. Therefore, we have

$$m_1 V_{1i} + m_2 V_{2i} = m_1 V_{1f} + m_2 V_{2f} \tag{9.15}$$

$$\frac{1}{2} m_1 V_{1i}^{\,2} + \frac{1}{2} m_2 V_{2i}^{\,2} = \frac{1}{2} m_1 V_{1f}^{\,2} + \frac{1}{2} m_2 V_{2f}^{\,2} \tag{9.16}$$

Let us cancel ½ in Equation 9.16 and rewrite it as

$$m_1 (V_{1i}^{\,2} - V_{1f}^{\,2}) = m_2 (V_{2f}^{\,2} - V_{2i}^{\,2})$$

And then factor both sides:

$$m_1 (V_{1i} - V_{1f})(V_{1i} + V_{1f}) = m_2 (V_{2f} - V_{2i})(V_{2f} + V_{2i}) \tag{9.17}$$

Next, let us separate the terms containing m_1 and m_2 in Equation 9.15 to obtain

$$m_1(V_{1i} - V_{1f}) = m_2(V_{2f} - V_{2i}) \tag{9.18}$$

To obtain our final result, we divide Equation 9.17 by Equation 9.18 and obtain

$$(V_{1i} + V_{1f}) = (V_{2f} + V_{2i})$$
$$V_{1i} - V_{2i} = -(V_{1f} - V_{2f}) \tag{9.19}$$

This equation, in combination with Equation 9.15, can be used to solve problems dealing with elastic collisions. According to Equation 9.19, the *relative* velocity of the two particles before collision, $V_{1i} - V_{2i}$, equal the negative of their relative velocity after the collision, $-(V_{1f} - V_{2f})$.

If the masses and initial velocities of both particles are know. Equations 9.15 and 9.19 can be solved for the final velocities in terms of the initial velocities:

From Equation 9.19:

$$V_{2f} = V_{1i} + V_{1f} - V_{2i}$$

Substitute the above equation into Equation 9.15:

$$m_1V_{1i} + m_2V_{2i} = m_1V_{1f} + m_2(V_{1i} + V_{1f} - V_{2i})$$
$$m_1V_{1i} + m_2V_{2i} = m_1V_{1f} + m_2V_{1i} + m_2V_{1f} - m_2V_{2i}$$
$$(m_1 - m_2)V_{1i} + 2m_2V_{2i} = (m_1 + m_2)V_{1f}$$

$$V_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)V_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right)V_{2i} \tag{9.20}$$

Again, From Equation 9.19:

$$V_{1f} = V_{2f} + V_{2i} - V_{1i}$$

Substitute the above equation into Equation 9.15:

$$m_1V_{1i} + m_2V_{2i} = m_1(V_{2f} + V_{2i} - V_{1i}) + m_2V_{2f}$$
$$m_1V_{1i} + m_2V_{2i} = m_1V_{2f} + m_1V_{2i} - m_1V_{1i} + m_2V_{2f}$$
$$2m_1V_{1i} + (m_2 - m_1)V_{2i} = (m_1 + m_2)V_{2f}$$

$$V_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right)V_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)V_{2i} \tag{9.21}$$

Let us consider some special cases. If $m_1 = m_2$, then Equation 9.20 and 9.21 show us that $V_{1f} = V_{2i}$ and $V_{2f} = V_{1i}$. That is, the particles exchange velocities if they have equal masses.

If particle 2 is initially at rest, then $V_{2i} = 0$, and Equation 9.20 and 9.21 become

$$V_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) V_{1i} \tag{9.22}$$

$$V_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) V_{1i} \tag{9.23}$$

If m_1 is much greater than m_2 and $V_{2i} = 0$, we see from Equations 9.22 and 9.23 that $V_{1f} \approx V_{1i}$ and $V_{2f} \approx 2V_{1i}$. That is, when a very heavy particle collides a head-on with a very light one that is initially at rest, the heavy particle continues its motion unaltered after the collision and the light particle rebounds with a speed equal to about twice the initial speed of the heavy particle.

If m_2 is much greater than m_1 and the particle 2 is initially at rest, then $V_{2f} \approx 0$. That is, when a very light particle collides head-on with a very heavy particle that is initially at rest, the light particle has its velocity reversed and the heavy one remain approximately at rest.

9.4 Two-Dimensional Collisions

For two dimension collisions, we obtain two component equations for conservation of momentum:

$$m_1 V_{1ix} + m_2 V_{2ix} = m_1 V_{1fx} + m_2 V_{2fx}$$

$$m_1 V_{1iy} + m_2 V_{2iy} = m_1 V_{1fy} + m_2 V_{2fy}$$

Where we use three subscripts in these equations to represent, respectively, (1) the identification of the object, (2) initial and final values, and (3) the velocity component.

Let us consider a two dimensional problem in which particle 1 of mass m_1 collides with particle 2 of mass m_2, where particle 2 is initially at rest, as in Figure 9.6. After collision, particle 1 moves at angle θ with respect to horizontal and particle 2 moves at an angle Φ with respect to the horizontal. This is called *glancing* collision. Applying the law of conservation of momentum in component form and noting that the initial y component of the momentum of the two-particle system is zero, we obtain

$$m_1 V_{1i} = m_1 V_{1f} \cos\theta + m_2 V_{2f} \cos\phi \tag{9.24}$$

$$0 = m_1 V_{1f} \sin\theta - m_2 V_{2f} \sin\phi \tag{9.25}$$

Where, the minus sign in equation 9.25 comes from the fact that after the collision, particle 2 has a y component of velocity that is downward.

If the collision is elastic, we can also use equation 9.16 (conservation of kinetic energy) with $V_{2i} = 0$ to give

$$\frac{1}{2} m_1 V_{1i}^{\,2} = \frac{1}{2} m_1 V_{1f}^{\,2} + \frac{1}{2} m_2 V_{2f}^{\,2} \tag{9.26}$$

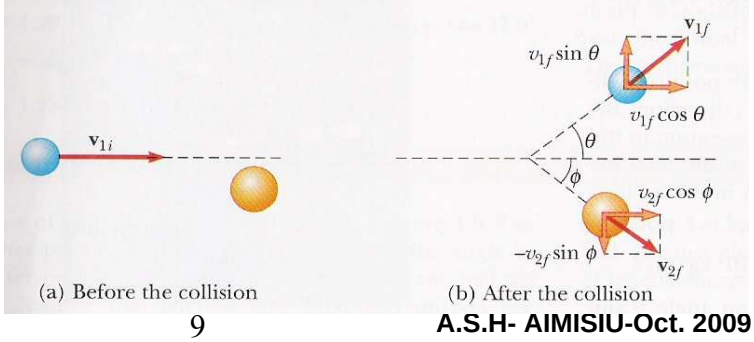


Figure 9.6 An elastic glancing collision between two particles.

Knowing the initial speed of particle 1 and both

masses, we are left with four unknowns (V_{1f}, V_{2f}, θ, and Φ). Because we have only three equations, one of the four remaining quantities must be given if we are to determine the motion after the collision from conservation principles alone.
If the collision is inelastic, kinetic energy is *not* conserved and equation 9.26 does *not* apply.

9.5 The Centre of Mass

In this section we describe the overall motion of a mechanical system in terms of a special point called the **centre of mass** of the system. The mechanical system can be either a group of particles, such as a collection of atoms in a container. If the resultant external force on the system is $\Sigma\mathbf{F}_{ext}$ and the total mass of the system is M, the centre of mass moves with an acceleration given by

$$\mathbf{a} = \Sigma\mathbf{F}_{ext}/M$$

The centre of mass of the system consist of a pair of particles is located somewhere on the line joining the two particles and is closer to the particle having the larger mass. If a single force is applied at a point on the rod somewhere between the centre of mass and the less massive particle, the system rotates clockwise (see Figure 9.7a). If the force is applied at a point on the rod somewhere between the centre of mass and the more massive particle, the system rotates counterclockwise (see Figure 9.7b). If the force is applied at the centre of mass, the system moves in the direction of **F** without rotating (see Figure 9.7c).

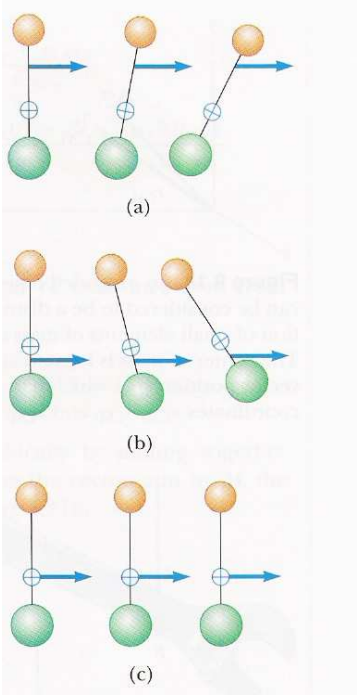


Figure 9.7 Two particles of unequal mass are connected by a light, rigid rod.

The centre of mass of the pair of particles described in Figure 9.8 is located on the x axis and lies somewhere between the particles. Its x coordinate is giving by

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \qquad (9.27)$$

If $x_1 = 0$, $x_2 = d$, and $m_2 = 2m_1$, we find that $x_{CM} = \frac{2}{3}d$.
That is, the centre of mass lies closer to the more massive particle. If the two masses are equal, the centre of mass lies midway between the particles.
The x coordinates of the centre of mass of n particles is defined to be

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ... + m_n x_n}{m_1 + m_2 + m_3 + ... + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M} \qquad (9.28)$$

Where x_i is the x coordinate of the ith particle, and $M \equiv \sum_i m_i$. The y and z coordinates of the centre of mass are similarly defined by the equations

$$y_{CM} = \frac{\sum_i m_i y_i}{M} \qquad \text{and} \qquad z_{CM} = \frac{\sum_i m_i z_i}{M} \tag{9.29}$$

The centre of mass can also be located by its position vector $\vec{r}_{CM}$

$$\vec{r}_{CM} = x_{CM}\vec{i} + y_{CM}\vec{j} + z_{CM}\vec{k} = \frac{\sum_i m_i x_i \vec{i} + \sum_i m_i y_i \vec{j} + \sum_i m_i z_i \vec{k}}{M}$$

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M} \tag{9.30}$$

Where $\vec{r}_i$ is the position vector or the *i*th particle, defined by:

$$\vec{r}_i = x_i\vec{i} + y_i\vec{j} + z_i\vec{k}$$

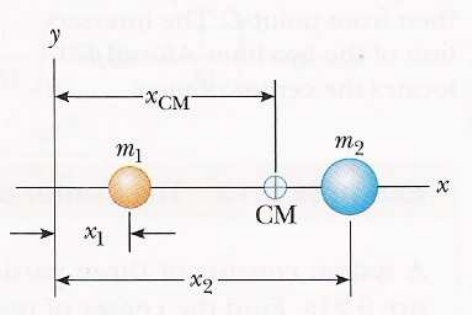


Figure 9.8 The centre of mass of two particles of unequal mass on the *x* axis is located at x_{CM}, a point between the particles, closer to the one having the larger mass.

For an extended object as a system containing a large number of particles (Figure 9.9). The particle separation is very small, and so object can be considered to have a continuous mass contribution. By dividing the object into elements of mass Δm_i with coordinates x_i, y_i, z_i, we see that the *x* coordinate of the centre of mass is approximately

$$x_{CM} \approx \frac{\sum_i x_i \Delta m_i}{M}$$

Which similar expression for y_{CM} and z_{CM}. If we let the number of elements *n* approach infinity, then x_{CM} is given precisely. In this limit, we replace the sum by integral and Δm_i by the element *dm*:

$$x_{CM} = \lim_{\Delta m_i \to o} \frac{\sum_i x_i \Delta m_i}{M} = \frac{1}{M}\int x\,dm \tag{9.31}$$

Likewise, for y_{CM} and z_{CM} we obtain

$$y_{CM} = \frac{1}{M}\int y\,dm \quad \text{and} \quad z_{CM} = \frac{1}{M}\int z\,dm \tag{9.32}$$

We can express the vector position of the centre of mass of an extended object in the form

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\,dm$$

The centre of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry. For example, the centre of mass of a uniform rod lies in the rod, midway between its ends. The centre of mass of a sphere or a cube lies at its geometric centre.

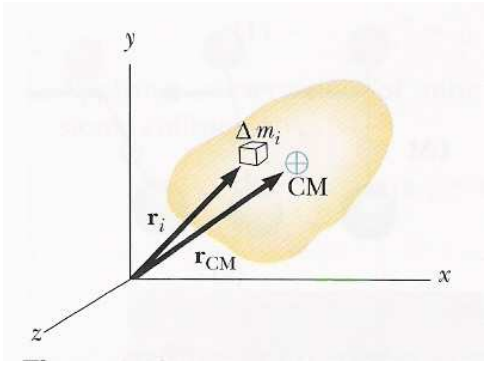


Figure 9.9 An extended object can be considered to be a distribution of small elements of mass Δm_i. The centre of mass is located at the vector position, which has coordinates x_{CM}, y_{CM}, and z_{CM}.

The centre of mass of irregularly shaped object such as a wrench can be determined by suspending from one point and then from other point. In Figure 9.10, a wrench is hung from point *A*, and a vertical line *AB* is drawn when the wrench has stopped swinging. The wrench is then hung from point *C*, and a second vertical line *CD* is drawn. The centre of mass is halfway through the thickness of the wrench, under the intersection of these two lines. In general, if the wrench is hung freely from any point, the vertical line through this point must pass through the centre of mass.
The net effect of all gravitational forces is equivalent to the effect of a single force $M\mathbf{g}$ acting through a special point, called the **centre of gravity.** If **g** is constant over the mass distribution, then the centre of gravity coincides with the centre of mass.

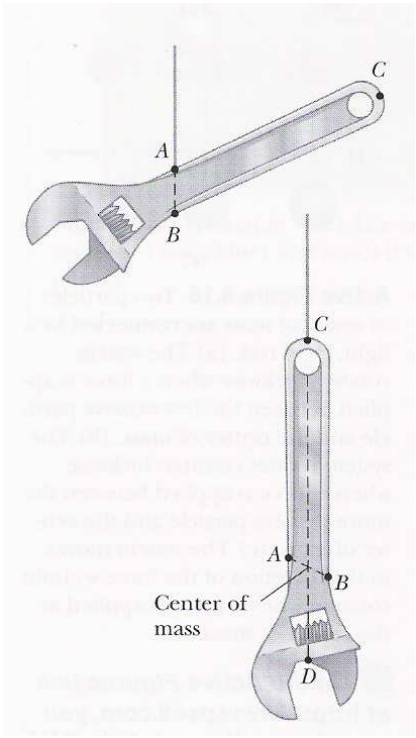


Figure 9.10 An experimental technique for determining the centre of mass of a wrench. The wrench is hung freely from point *A* and then from point *C*. The intersection of the two lines *AC* and *CD* locates the centre of mass.

9.6 Motion of a System of Particles

Assuming *M* remains constant for a system of particles, that is, no particles enter or leave the system, (and by using Equation 9.30; $\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{M}$) we obtain the following expression for the velocity of the **centre of mass** of the system:

$$\vec{V}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{\sum_i m_i \vec{V}_i}{M} \qquad (9.34)$$

Where $\vec{V}_i$ is the velocity of the *i*th particle. Rearranging Equation 9.34 gives

$$M\vec{V}_{CM} = \sum_i m_i \vec{V}_i = \sum_i p_i = p_{tot} \qquad (9.35)$$

Therefore, we conclude that the **total linear momentum of the system equals the total mass multiplied by the velocity of the centre of mass.**

If we now differentiate Equation 9.34 with respect to time, we obtain the **acceleration of the centre of mass** of the system:

$$\vec{a}_{CM} = \frac{d\vec{V}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{V}_i}{dt} = \frac{1}{M}\sum_i m_i a_i \qquad (9.36)$$

Rearranging this expression and using Newton's second law, we obtain

$$M\vec{a}_{CM} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i \qquad (9.37)$$

Where $\vec{F}_i$ is the net force on particle *i*.

The forces on any particle in the system may include both external forces and internal forces. By Newton's second law, the sum over all internal forces in Equation 9.37, they cancel in pairs and we find that the net force on the system is caused only by external force. Thus, we can write Equation 9.37 in the form

$$\sum \vec{F}_{ext} = M\vec{a}_{CM} \qquad (9.38)$$

That is, **the net external force on a system of particles equals the total mass of the system multiplied by the acceleration of the centre of mass.**

Finally, we see that if the net external force is zero, then from Equation 9.38 it fellows that

$$M\vec{a}_{CM} = M\frac{d\vec{V}_{CM}}{dt} = 0$$

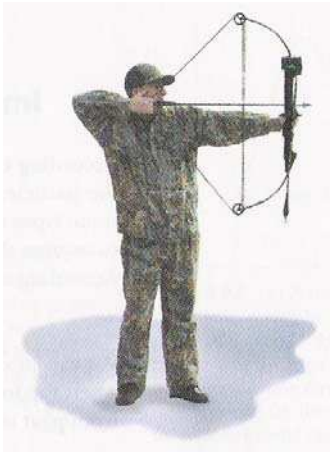
so that

$$M\vec{V}_{CM} = P_{tot} = cons\tan t \qquad (\text{when } \sum F_{ext} = 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (9.39)$$

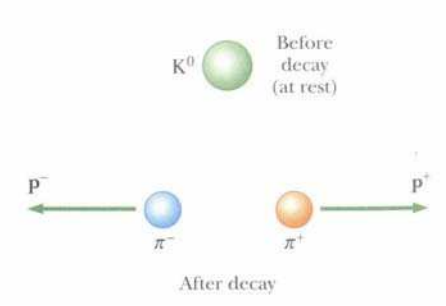
That is, the total linear momentum of a system of particles is conserved if no net external force acting on the system.

9.7 Exercises

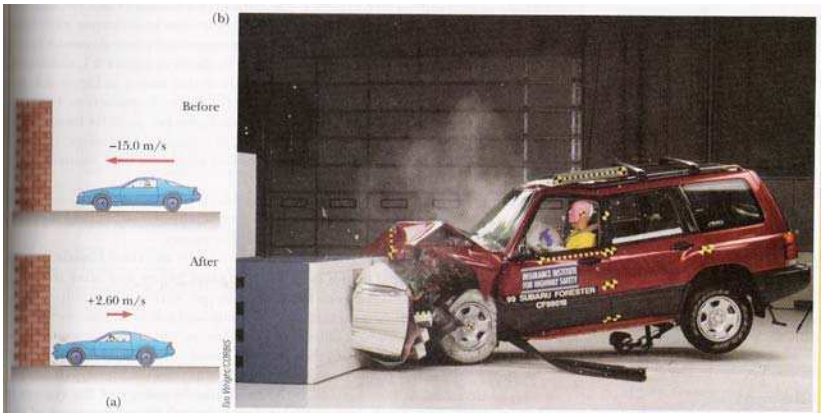
Example 1



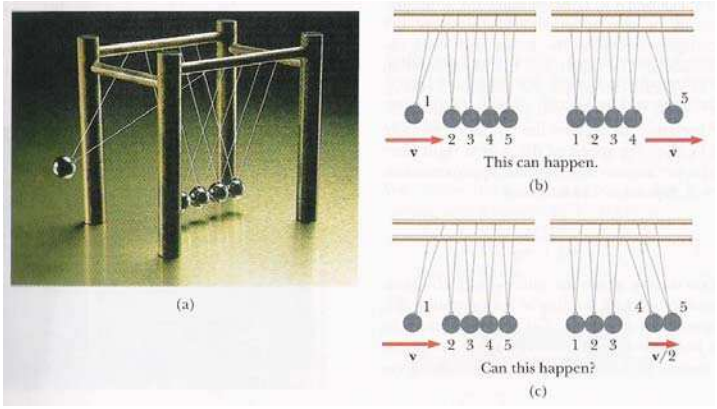
Example 2



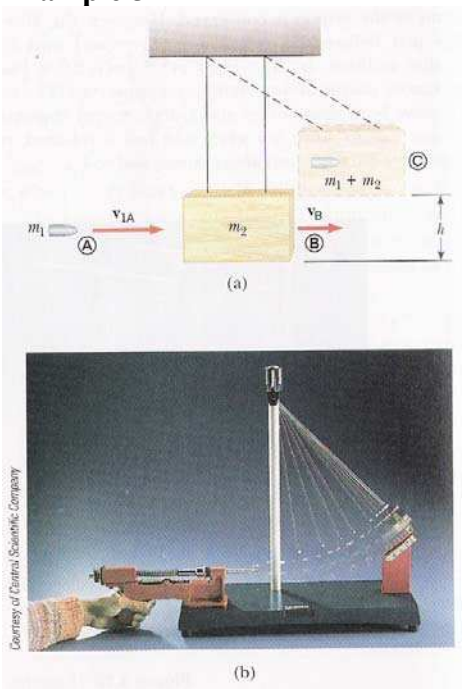
Example 3



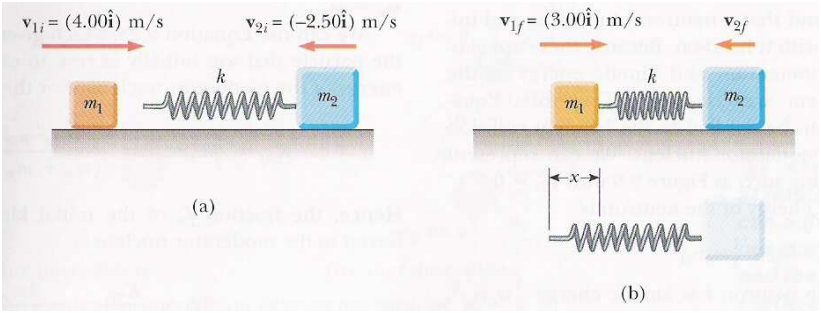
Examle 4



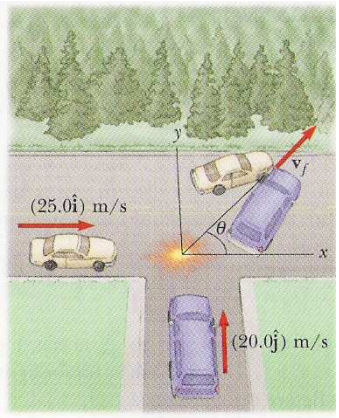
Example 5



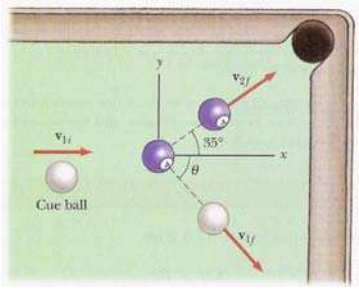
Example 6



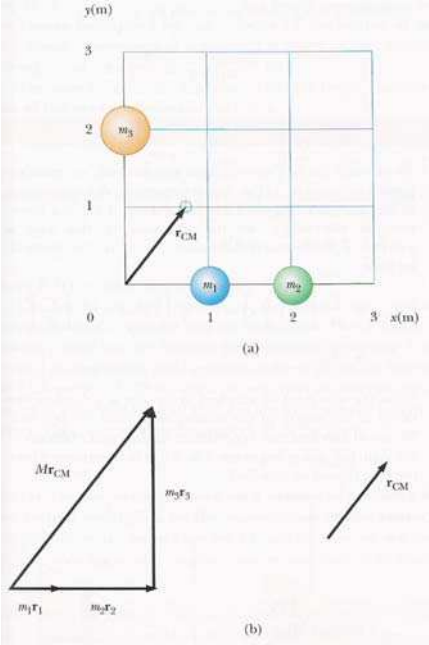
Example 7



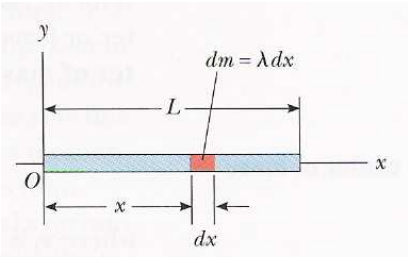
Example 8



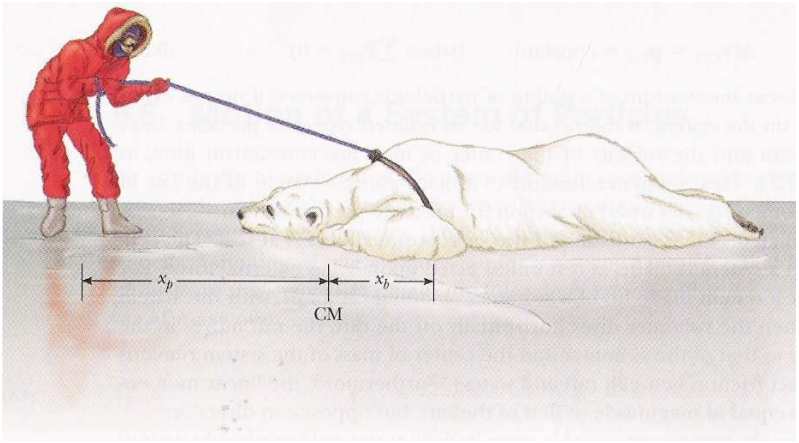
Example 9



Example 10



Example 11



9.8 Homework

9.8.1 A 3 kg particle has a velocity of (3iˆ - 4jˆ) m/s. (a) Find its x and y components of momentum. (b) Find the magnitude and direction of its momentum.

9.8.2 (a) A particle of mass m moves with momentum p. Show that the kinetic energy of the particle is $K = p^2/2m$. (b) Express the magnitude of the particle's momentum in terms of its kinetic energy and mass.

9.8.3 An estimated force–time curve for a baseball struck by a bat is shown in the Figure below. From this curve, determine (a) the impulse delivered to the ball, (b) the average force exerted on the ball, and (c) the peak force exerted on the ball.

9.8.4 Two blocks are free to slide along the frictionless wooden track ABC shown in the Figure below. The block of mass $m_1 = 5$ kg is released from A. Protruding from its front end is the north pole of a strong magnet, repelling the north pole of an identical magnet embedded in the back end of the block of mass $m_2 = 10$ kg, initially at rest. The two blocks never touch. Calculate the maximum height to which m_1 rises after the elastic collision.

9.8.5 Two shuffleboard disks of equal mass, one orange and the other yellow, are involved in an elastic, glancing collision. The yellow disk is initially at rest and is struck by the orange disk moving with a speed of 5 m/s. After the collision, the orange disk moves along a direction that makes an angle of 37° with its initial direction of motion. The velocities of the two disks are perpendicular after the collision. Determine the final speed of each disk.

9.8.6 Four objects are situated along the y axis as follows: a 2 kg object is at +3 m, a 3 kg object is at +2.5 m, a 2.5 kg object is at the origin, and a 4 kg object is at - 0.5 m. Where is the center of mass of these objects?

9.8.7 A 2 kg particle has a velocity (2iˆ - 3jˆ) m/s, and a 3 kg particle has a velocity (1iˆ + 6jˆ) m/s. Find (a) the velocity of the center of mass and (b) the total momentum of the system.

Chapter 10

Rotation of a Rigid Object About a Fixed Axis

In dealing with a rotating object, analysis is greatly simplified by assuming that the object is rigid. A **rigid object** is one that is "nondeformable" that is, the relative location of all particles of which the object is composed remains constant.

10.1 Angular Position, velocity, and Acceleration

Figure 10.1 illustrates an overhead view of a rotating disc. The disc is rotating about a fixed axis through *O*. A particle *P* is at a fixed distance *r* from the origin and rotates about it in a circle of radius *r*. The position of *P* with its polar coordinates is (*r, θ*), where *θ* is measured in *counterclockwise* from some reference line as shown in Figure 10.1a. As the particle moves along the circle from the reference line (*θ* = 0), it moves through an arc of length *s*, as in Figure 10.1b. The arc length *s* is related to the angle *θ* through the relationship

$$s = r\theta \tag{10.1a}$$

$$\theta = \frac{s}{r} \tag{10.1b}$$

The artificial unit of *θ* is **radian** (rad), where: **one radian is the angle subtended by the arc length equal to the radius of the arc.**

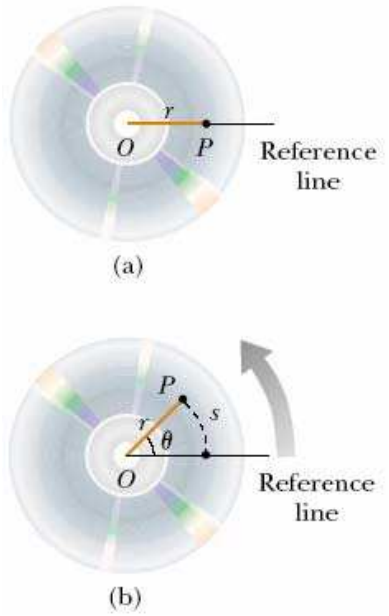


Figure 10.1 A compact disc rotating about a fixed axis through *O* perpendicular to the plane of the figure.

Because the circumference of the circle is $2\pi r$, it follows Equation 10.1b that 360 ° correspond to an angle of $(2\pi r/r)$ rad = 2π rad. Hence 1 rad = $360^o/2\pi \approx$ 57.3 °. To convert an angle in degree to an angle in radians, we use the fact that π rad =180 °, or

$$\theta(rad) = \frac{\pi}{180^o}\theta(\text{deg})$$

In Figure 10.1, every particle on the object rotates through the same angle *θ*. Thus, **we can associate *θ* with the entire rigid object as well as with an individual particle.** This allows us to define the *angular position* of a rigid object in its rotational motion, which is the angle *θ* between the reference line on the object and the fixed reference line in space, which is choose to be the *x* axis.

As the particle in our rigid object travels from position (A) to position (B) in a time interval Δt as in Figure 10.2, the reference line of the length *r* sweeps out an angle

$\Delta\theta = \theta_f - \theta_i$. The quantity $\Delta\theta$ is defined as the **angular displacement** of the rigid object:

$$\Delta\theta \equiv \theta_f - \theta_i$$

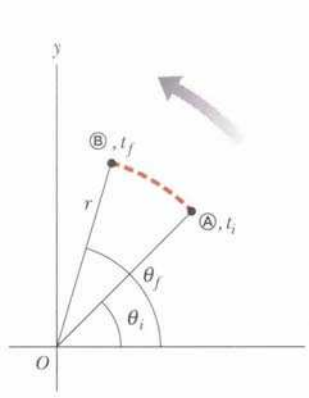


Figure 10.2 A particle on a rotating rigid object moves from (A) to (B) along the arc of a circle. In the time interval $\Delta t = t_f - t_i$, the radius vector moves through an angular displacement $\Delta\theta = \theta_f - \theta_i$.

We define the **average angular speed** ϖ (omega) as**:**

$$\varpi = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t} \qquad (10.2)$$

The **instantaneous angular speed** ω is defined as the limit of the ratio $\Delta\theta/\Delta t$ as Δt approaches zero:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt} \qquad (10.3)$$

Angular speed has units of radians per second (rad/s) which can be written as second^{-1} (s^{-1}) because the radians are not dimensional. We taking ω to be positive when, θ is increasing (counterclockwise) and negative when θ is decreasing (clockwise).

If the instantaneous angular speed of an object changes from ω_i to ω_f in the time interval Δt, the object has an angular acceleration. The **average angular acceleration** $\overline{\alpha}$ (alpha) of rotating rigid object is defined as the ratio of the change in the angular speed to the time interval Δt during which the change in angular speed occurs:

$$\overline{\alpha} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta\omega}{\Delta t} \qquad (10.4)$$ Average angular acceleration

The **instantaneous angular acceleration** is defined as the limit of the ratio $\Delta\omega/\Delta t$ as Δt approaches zero:

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} \qquad (10.5)$$ Instantaneous angular acceleration

Angular acceleration has units of (rad/s^2), or just (s^{-2}). Note that α is positive when a rigid object rotating counterclockwise is speeding up or when a rigid object rotating clockwise is slowing down some time interval.

Angular position (θ), angular speed (ω), and angular acceleration (α) are analogous to linear position (x), linear speed (v), and linear acceleration (a).

Strictly speaking, ω and α are the magnitude of angular velocity and angular acceleration vector $\vec{\omega}$ and $\vec{\alpha}$, respectively, and they always be positive. If an object rotates in the xy plane as in Figure 10.1, the direction of $\vec{\omega}$ is out of the plane of the diagram when the rotation is counterclockwise and into the plane of the diagram when the rotation is clockwise. To illustrate this convection, it is convenient to use the *right-hand rule* demonstrated in Figure 10.3.

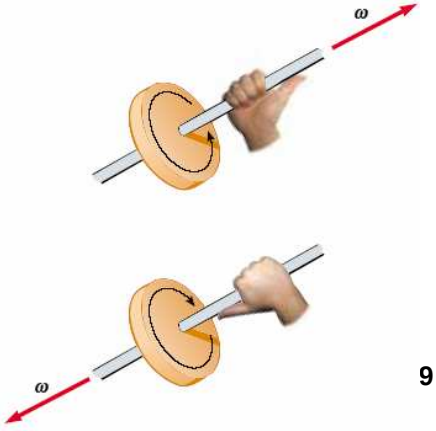


Figure 10.3 The right hand rule for determining the direction of the angular velocity vector.

10.2 Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

If we write Equation 10.5 in the form $d\omega = \alpha dt$, and let $t_i = 0$ and $t_f = t$, integration this expression directly gives

$$\omega_f = \omega_i + \alpha t \qquad \text{(for constant } \alpha\text{)} \qquad (10.6)$$

Where ω_i is the angular speed of the rigid object at time $t = 0$. Substituting Equation 10.6 into Equation 10.3 ($\omega_f = \frac{d\theta}{dt}$) and integrating once more, we obtain

$$\omega_i + \alpha t = \frac{d\theta}{dt}$$

$$\int_0^t \omega_i dt + \int_0^t \alpha t dt = \int_{\theta_i}^{\theta_f} d\theta$$

$$\omega t + \frac{1}{2}\alpha t^2 = \theta_f - \theta_i$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \qquad \text{(for constant } \alpha\text{)} \qquad (10.7)$$

Where θ_i is the angular position of the rigid object at time $t = 0$. If we eliminate t from Equation 10.6,

$$t = \frac{\omega_f - \omega_i}{\alpha}$$

Substitute t into Equation (10.7)

$$\theta_f = \theta_i + \omega_i\left(\frac{\omega_f - \omega_i}{\alpha}\right) + \frac{1}{2}\alpha\left(\frac{\omega_f - \omega_i}{\alpha}\right)^2$$

$$2\alpha(\theta_f - \theta_i) = 2\omega_i\omega_f - 2\omega_i^2 + \omega_f^2 - 2\omega_i\omega_f + \omega_i^2$$

$$2\alpha(\theta_f - \theta_i) = -\omega_i^2 + \omega_f^2$$

$$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i) \qquad \text{(for constant } \alpha\text{)} \qquad (10.8)$$

If we eliminate α from Equation 10.6, we obtain

$$\alpha = \frac{\omega_f - \omega_i}{t}$$

Substitute into Equation (10.7)

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\left(\frac{\omega_f - \omega_i}{t}\right)t^2$$

$$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t \qquad \text{(for constant } \alpha\text{)} \qquad (10.9)$$

Note that these kinematics expressions for rotational motion under constant angular acceleration are of the same mathematical form as those for linear motion under constant linear acceleration. They can be generated from the equations for linear

motion by making substitutions $x \to \theta$, $v \to \omega$, and $a \to \alpha$. Table 10.1 compares the kinematics equations for rotational and linear motion.

Table 10.1

Kinematic Equations for Rotational and Linear Motion Under Constant Acceleration

Rotational Motion About Fixed Axis	Linear Motion
$\omega_f = \omega_i + \alpha t$	$v_f = v_i + at$
$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$	$x_f = x_i + v_i t + \frac{1}{2}at^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\theta_f - \theta_i)$	$v_f^2 = v_i^2 + 2a(x_f - x_i)$
$\theta_f = \theta_i + \frac{1}{2}(\omega_i + \omega_f)t$	$x_f = x_i + \frac{1}{2}(v_i + v_f)t$

10.3 Angular and Linear Quantities

When a rigid object rotates about a fixed axis, as in Figure 10.4, **every particle of the object moves in a circle whose centre is the axis of rotation.** Because point *P* in Figure 10.4 moves in a circle, the linear velocity vector **v** is always tangent to the circular path and hence is called *tangential velocity.* The tangential speed at point *P* is $v = ds/dt$. Recalling $s = r\theta$ (Equation 10.1a) and noting that *r* is constant, we obtain

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt}$$

Because, $d\theta/dt = \omega$, we see that

$$v = r\omega \tag{10.10}$$

Note that, not every point on the rigid object has the same *tangential* speed because *r* is not the same for all points on the object. Equation 10.10 shows that *v* increases as one moves outward from the centre of rotation.

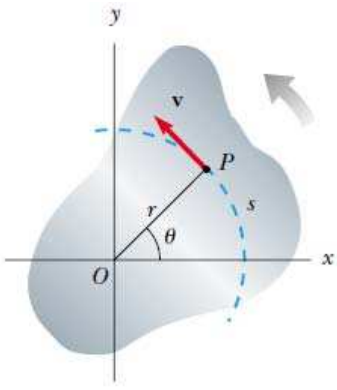


Figure 10.4 As a rigid object rotates about the fixed axis through *O*, the point *P* has a tangential velocity **v** always tangent to the circular path of radius *r*.

We can relate the angular acceleration of rotating rigid object to the tangential acceleration of the point *P* by taking the time derivative of *v*:

$$a_t = \frac{dv}{dt} = r\frac{d\omega}{dt}$$

$$a_t = r\alpha \tag{10.11}$$

In section 4.4 we found that the radial acceleration a_r of magnitude v^2/r directed toward the centre of rotation (Fig 10.5). Because we have $v = r\omega$, $a_r = v^2/r$, and $a_r = a_c$, so we can express the centripetal acceleration in term of angular speed as

$$a_c = \frac{v^2}{r} = r\omega^2 \qquad (10.12)$$

The total linear acceleration of this point is $\mathbf{a} = a_t + a_r$, where the magnitude of $\boldsymbol{a}_r$ is the centripetal acceleration a_c. Because **a** is a vector having a radial and tangential component, the magnitude of **a** at point *P* on the rotating rigid object is

$$a = \sqrt{a_t^{\,2} + a_r^{\,2}} = \sqrt{r^2\alpha^2 + r^2\omega^4} = r\sqrt{\alpha^2 + \omega^4} \qquad (10.13)$$

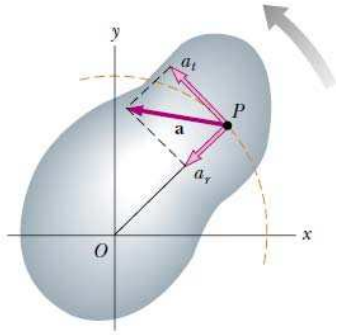


Figure 10.5 As a rigid object rotates about the fixed axis through *O*, the point *P* experiences a tangential component of linear acceleration a_t and a radial component of linear acceleration a_r. The total linear acceleration of this point is $\mathbf{a} = a_t + a_r$.

10.4 Rotating Kinetic Energy

Let us consider an object as a collection of particles and assume that it rotates about a fixed *z* axis with angular speed ω. Figure 10.6 shows the rotating object and identifies one particle on the object located at distance r_i from the rotation axis.

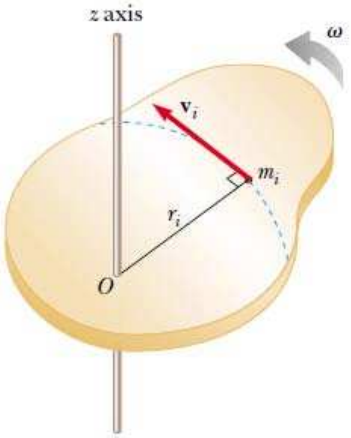


Figure 10.6 A rigid object rotating about the *z* axis with angular speed ω.

If the mass of the *i*th particle is m_i and its tangential speed is v_i, its kinetic energy is

$$K_i = \frac{1}{2} m_i v_i^{\,2}$$

The *total* kinetic energy of the rotating rigid object "K_R" is the sum of the kinetic energies of the individual particles:

$$K_R = \sum_i K_i = \sum_i \frac{1}{2} m_i v_i^{\,2}$$

By using equation 10.10, we get:

$$K_R = \frac{1}{2} \sum_i m_i r_i^{\,2} \omega^2$$

We can write this expression in the form

$$K_R = \frac{1}{2} \left(\sum_i m_i r_i^{\,2} \right) \omega^2 \qquad (10.14)$$

We simplify this expression by defining the quantity in parentheses as the ***moment of inertia I***:

$$I = \sum_i m_i r_i^2 \tag{10.15}$$

From the definition of moment of inertia, we see that it has dimensions of ML^2 ($kg.m^2$ in SI unit). With this notation, Equation 10.14 becomes

$$K_R = \frac{1}{2} I\omega^2 \tag{10.16}$$

We refer to the quantity ½ $I\omega^2$ as **rotational kinetic energy**. The mathematical form of the above equation is convenient when we are dealing with rotational motion, provided we know how to calculate I. The quantities I and ω in rotational motion are analogues to m and v in linear motion, respectively. The moment of inertia is a measure of the resistance of an object to changes in its rotational motion, just as mass is a measure of the tendency of an object to resist changes in its linear motion.

10.5 Calculation of Moment of Inertia

We can evaluate the moment of inertia of an extended rigid object by imagining the object to be divided into many small volume elements, such of which has mass Δm_i. We use the definition $I = \sum_i r_i^2 \Delta m_i$ and take the limit of this sum as $\Delta m_i \rightarrow 0$. In this limit, the sum becomes an integral over the volume of the object:

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm \tag{10.17}$$

From Equation 1.1

$$\rho \equiv m/V$$

Where ρ is the density of the object and V is the volume. From this equation, the mass of small element is $dm = \rho dV$. Substituting this result into Equation 10.17 gives

$$I = \int \rho r^2 dV$$

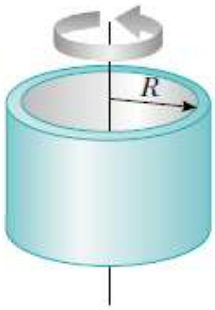
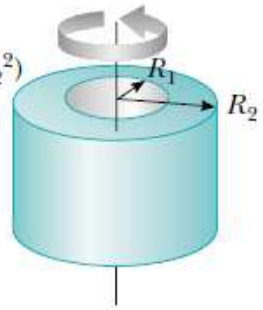
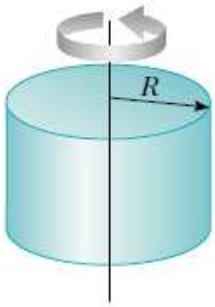
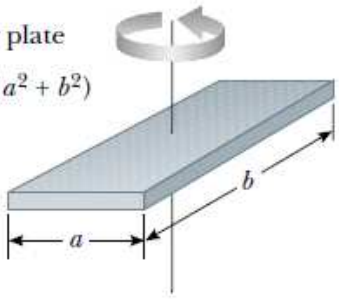
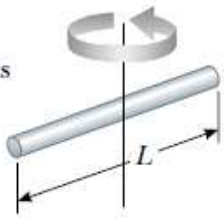
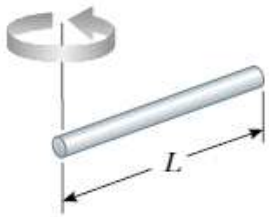
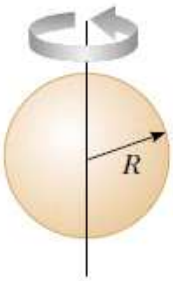
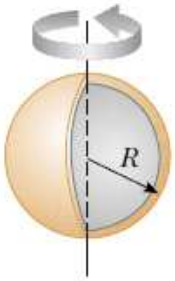
If the object is homogeneous, the ρ is constant and the integral can be evaluated for a known geometry. If ρ is not constant, then the variation with position must be known to complete the integration.

The density $\rho = m/V$ sometimes is referred to as *volumetric mass density*. For a sheet of uniform thickness t, we can define a *surface mass density* $\sigma = \rho t$, which represent mass per unit area. Finally, when mass is distributed along a rod of uniform cross-sectional area A, we sometimes use *linear mass density* $\lambda = M/L = \rho A$, which is the *mass per unit length*.

Table 10.2 gives the moments of inertia for a number of objects about specific axes. The moments of inertia of rigid objects with simple geometry (high symmetry) are relatively easy to calculate provided the rotation axis coincides with an axis of symmetry. The moment of inertia about an axis through the centre of mass of an object is I_{CM}.

Table 10.2

Moments of Inertia of Homogeneous Rigid Objects with Different Geometries

Hoop or thin cylindrical shell $I_{CM} = MR^2$	
Hollow cylinder $I_{CM} = \frac{1}{2}M(R_1^2 + R_2^2)$	
Solid cylinder or disk $I_{CM} = \frac{1}{2}MR^2$	
Rectangular plate $I_{CM} = \frac{1}{12}M(a^2 + b^2)$	
Long thin rod with rotation axis through center $I_{CM} = \frac{1}{12}ML^2$	
Long thin rod with rotation axis through end $I = \frac{1}{3}ML^2$	
Solid sphere $I_{CM} = \frac{2}{5}MR^2$	
Thin spherical shell $I_{CM} = \frac{2}{3}MR^2$	

10.6 Torque

When a force is exerted on a rigid object pivoted about an axis, the object tends to rotate about that axis. The tendency of a force to rotate an object about some axis is measured by a vector quantity called **torque** τ (Greek tau). Torque is a vector, but we will consider only its magnitude here.

Consider the wrench pivoted on the axis through O in Figure 10.7. The applied force **F** acts at an angle ϕ to the horizontal. We define the magnitude of the torque associated with the force **F** by the expression

$$\tau \equiv rF\sin\phi = Fd \qquad (10.19)$$

Where r is the distance between the pivot point and the point of application of **F** and d is the distance from the pivot point to the line of action of **F**. From the right triangle in Figure 10.7 that has the wrench as its hypotenuse, we see that $d = r\sin\phi$. The quantity d is called the moment arm (or *lever arm*) of **F**.

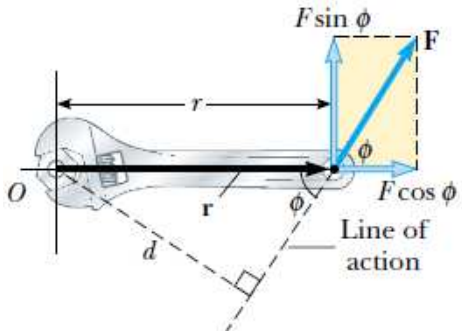


Figure 10.7 The force **F** has a greater rotating tendency about O as F increases and as the moment arm d decreases. The component $F\sin\phi$ tends to rotate the wrench about O.

In Figure 10.7, the only component of **F** that tends to cause rotation is $F\sin\phi$. The horizontal component $F\cos\phi$, because its line of action passes through O, has no tendency to produce rotation about an axis passing through O. From the definition of torque, we see that the rotating tendency increases as F increases and as d increases.

If two or more forces are acting on a rigid object, as in Figure 10.8, each tends to produce rotation about the axis O. In this example $\mathbf{F}_2$ tends to rotate the object clockwise and $\mathbf{F}_1$ tends to rotate it counterclockwise. The sign of torque is positive if the turning tendency is counterclockwise and is negative if the turning tendency is clockwise. In Figure 10.8, the torque resulting from $\mathbf{F}_1$ is positive and equal + F_1d_1; the torque from $\mathbf{F}_2$ is negative and equal to – F_2d_2. Hence the *net* torque about O is

$$\sum\tau = \tau_1 + \tau_2 = F_1d_1 - F_2d_2$$

Torque should not be confused with force. Torque has units of force time length "Newton.meters" in SI units.

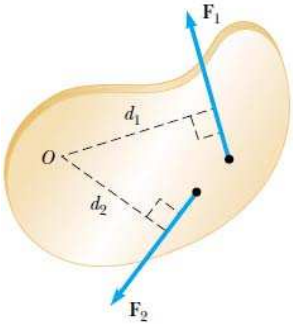


Figure 10.8 The force $\mathbf{F}_1$ tends to rotate the object counterclockwise about O, and $\mathbf{F}_2$ tends to rotate it clockwise.

10.7 The rigid object under a net torque

Consider a particle of mass m rotating in a circle of radius r under the influence of a tangential force $\mathbf{F}_t$ and a radial force $\mathbf{F}_r$, as shown in Figure 10.9. The tangential force provides a tangential acceleration $\mathbf{a}_t$, and

$$F_t = ma_t$$

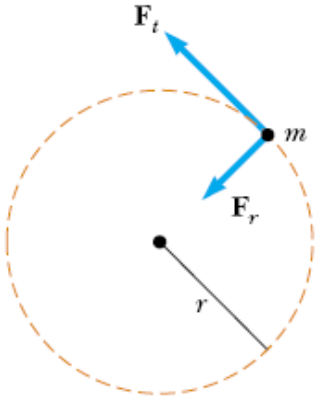


Figure 10.9 A particle rotating in a circle under the influence of a tangential force $\mathbf{F}_t$. A force $\mathbf{F}_r$ in the radial direction also must be present to maintain the circular motion.

The magnitude of the torque about the center of the circle due to $\mathbf{F}_t$ is

$$\tau = F_t\, r = (ma_t)r$$

Because the tangential acceleration is related to the angular acceleration through the relationship $a_t = r\alpha$ (Equation 10.11), the torque can be expressed as

$$\tau = (mr\alpha)r = (mr^2)\,\alpha$$

Recall from Equation 10.15 that mr^2 is the moment of inertia of the particle about the z axis passing through the origin, so that

$$\tau = I\,\alpha \qquad (10.20)$$

That is, **the torque acting on the particle is proportional to its angular acceleration**, and the proportionality constant is the moment of inertia. Note that $\tau = I\alpha$ is the rotational analog of Newton's second law of motion, $F = ma$.

Now let us extend this discussion to a rigid object of arbitrary shape rotating about a fixed axis, as in Figure 10.10.

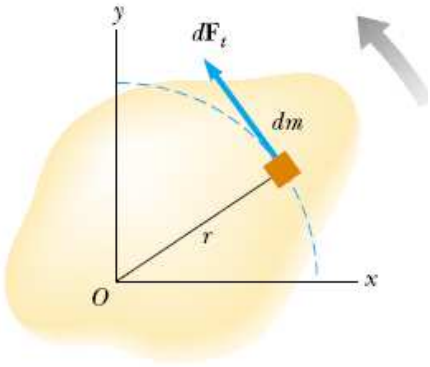


Figure 10.10 A rigid object rotating about an axis through O. Each mass element dm rotates about O with the same angular acceleration α, and the net torque on the object is proportional to α.

The object can be regarded as an infinite number of mass elements dm of infinitesimal size. If we impose a Cartesian coordinate system on the object, then each mass element rotates in a circle about the origin, and each has a tangential acceleration $\mathbf{a}_t$

produced by an external tangential force $d\mathbf{F}_t$. For any given element, we know from Newton's second law that

$$dF_t = (dm)a_t$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ acts about the origin and is given by

$$d\tau = rdF_t = a_t r dm$$

Because $a_t = r\alpha$, the expression for $d\tau$ becomes

$$d\tau = \alpha r^2 dm$$

We can integrate the above expression to obtain the net torque $\sum\tau$ about O due to the external forces:

$$\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm$$

Where α can be taken outside the integral because it is common to all mass elements. From Equation 10.17, we know that is the moment of inertia of the object about the rotation axis through O, and so the expression for $\sum\tau$ becomes

$$\sum\tau = I\alpha \qquad (10.21)$$

Note that this is the same relationship we found for a particle moving in a circular path (see Eq. 10.20). So, again we see that the net torque about the rotation axis is proportional to the angular acceleration of the object, with the proportionality factor being I. In view of the complex nature of the system, the relationship $\sum\tau = I\alpha$ is strikingly simple and in complete agreement with experimental observations.

10.6 Homework

Angular position, Velocity and Acceleration

10.6.1 During a certain period of time, the angular position of a swinging door is described by $\theta = 5 + 10t + 2t^2$, where θ is in radians and t is in seconds. Determine the angular position, angular speed, and angular acceleration of the door (a) at $t = 0$ (b) at $t = 3$ s.

Rotational Kinematics: Rotational Motion with Constant Angular Acceleration

10.6.2 A wheel starts from rest and rotates with constant angular acceleration to reach an angular speed of 12 rad/s in 3 s. Find (a) the magnitude of the angular acceleration of the wheel and (b) the angle in radians through which it rotates in this time.

Angular and Linear Quantities

10.6.4 A wheel 2 m in diameter lies in a vertical plane and rotates with a constant angular acceleration of 4 rad/s^2. The wheel starts at rest at $t = 0$, and the radius vector of a certain point P on the rim (إطار؛ حافة مستديره) makes an angle of 57.3° with the

horizontal at this time. At t = 2 s, find (a) the angular speed of the wheel, (b) the tangential speed and the total acceleration of the point P, and (c) the angular position of the point P.

Rotational Kinetic Energy

10.6.5) Rigid rods of negligible mass lying along the y axis connect three particles (Fig. P10.20). If the system rotates about the x axis with an angular speed of 2 rad/s, find (a) the moment of inertia about the x axis and the total rotational kinetic energy evaluated from 1/2 $I\omega^2$ and (b) the tangential speed of each particle and the total kinetic energy evaluated from $\sum_i \frac{1}{2} m_i v_i^2$.

Torque

10.6.7 The fishing pole in Figure P10.30/28 makes an angle of 20° with the horizontal. What is the torque exerted by the fish about an axis perpendicular to the page and passing through the fisher's hand?

Relationship between Torque and Angular Acceleration

10.6.8 A model airplane with mass 0.75 kg is tethered (قيدت) by a wire so that it flies in a circle 30 m in radius. The airplane engine provides a net thrust (يدفع) of 0.8 N perpendiculars to the tethering wire. (a) Find the torque the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the linear acceleration of the airplane tangent to its flight path.

Chapter 11

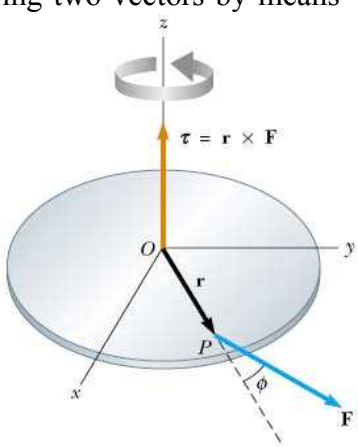
Angular Momentum

The angular momentum is a quantity that plays a key role in rotational dynamics. In analogy to the principle of conservation of linear momentum, we find that the angular momentum of a system is conserved if no external torques acts on the system.

11.1 The Vector Product and Torque

The angular momentum defined as the process of multiplying two vectors by means of the operation called the *vector product*.

Consider a force **F** acting on a rigid object at the vector position **r** (Fig. 11.1). As we saw in Section 10.6, the *magnitude* of the torque due to this force relative to the origin is $rF\sin ø$, where ø is the angle between **r** and **F**.



The axis about which **F** tends to produce rotation is perpendicular to the plane formed by **r** and **F**.

Figure 11.1 The torque vector τ lies in a direction perpendicular to the plane formed by the position vector **r** and the applied force vector **F**.

We can establish a mathematical relationship between τ, **r**, and **F** using a mathematical operation called the **vector product**, or **cross product**:

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{11.1}$$

For any two vectors **A** and **B**, the vector product **A** × **B** is defined as a third vector **C**, which has a magnitude of $AB \sin\theta$, where θ is the angle between **A** and **B**. That is, if **C** is given by

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{11.2}$$

Then its magnitude is

$$C = AB \sin\theta \tag{11.3}$$

The quantity $AB \sin\theta$ is equal to the area of the parallelogram formed by **A** and **B**, as shown in Figure 11.2. The *direction* of **C** is perpendicular to the plane formed by **A** and **B**, and the best way to determine this direction is to use the right-hand rule illustrated in Figure 11.2. The four fingers of the right hand are pointed along **A** and then "wrapped" into **B** through the angle θ. The direction of the upright thumb is the direction of **A** × **B** = **C**. Because of the notation, **A** × **B** is often read "**A** cross **B**"; hence, the term *cross product.*

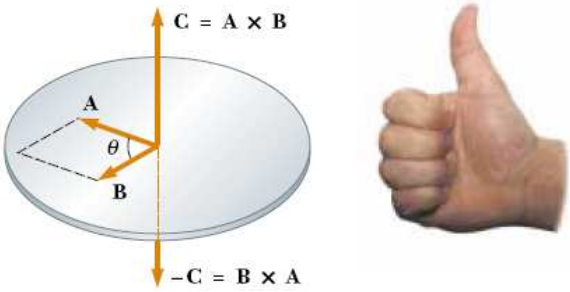


Figure 11.2 The vector product **A** × **B** is a third vector **C** having a magnitude $AB \sin\theta$ equal to the area of the parallelogram shown. The direction of **C** is perpendicular to the plane formed by **A** and **B**, and this direction is determined by the right-hand rule.

Some properties of the vector product that follow from its definition are as follows:

1. Unlike the scalar product, the vector product is *not* commutative.

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{11.4}$$

Therefore, if you change the order of the vectors in a cross product, you must change the sign.

2. If **A** is parallel to **B** ($\theta = 0°$ or 180°), then **A** x **B** = 0; therefore, it follows that

$$\mathbf{A} \times \mathbf{A} = 0.$$

3. If **A** is perpendicular to **B**, then $|\mathbf{A} \times \mathbf{B}| = AB$.

4. The vector product obeys the distributive law:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \tag{11.5}$$

Properties of the vector product

5. The derivative of the cross product with respect to some variable such as t is

$$\frac{d}{dt}(\vec{A}\times\vec{B}) = \frac{d\vec{A}}{dt}\times\vec{B} + \vec{A}\times\frac{d\vec{B}}{dt} \qquad (11.6)$$

The definition of unit vectors that the cross products of the rectangular unit vectors **i** , **j** , and **k** obey the following rules:

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0 \qquad (11.7a)$$

$$\mathbf{i}\times\mathbf{j} = -\,\mathbf{j}\times\mathbf{i} = \mathbf{k} \qquad (11.7b)$$

$$\mathbf{j}\times\mathbf{k} = -\,\mathbf{k}\times\mathbf{j} = \mathbf{i} \qquad (11.7c)$$

$$\mathbf{k}\times\mathbf{i} = -\,\mathbf{i}\times\mathbf{k} = \mathbf{j} \qquad (11.7d)$$

Cross products of unit vectors

Signs are interchangeable in cross products. For example,

$$\mathbf{A}\times(-\mathbf{B}) = -\,\mathbf{A}\times\mathbf{B}$$

and

$$\mathbf{i}\times(-\mathbf{j}) = -\,\mathbf{i}\times\mathbf{j}.$$

The cross product of any two vectors A and B can be expressed in the following determinant form:

$$\vec{A}\times\vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix}\vec{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}\vec{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}\vec{k}$$

Expanding these determinants gives the result

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\vec{i} - (A_xB_z - A_zB_x)\vec{j} + (A_xB_y - A_yB_x)\vec{k} \qquad (11.8)$$

Given the definition of the cross product, we can now assign a direction to the torque vector. If the force lies in the xy plane, as in Figure 11.1, the torque τ is represented by a vector parallel to the z axis. The force in Figure 11.1 creates a torque that tends to rotate the object counterclockwise about the z axis; thus the direction of τ is toward increasing z, and τ is therefore in the positive z direction. If we reversed the direction of **F** in Figure 11.1, then τ would be in the negative z direction.

11.2 Angular Momentum

In this chapter we will follow a procedure for problem-solving similar to that in chapter 9.

Consider a particle of mass m located at the vector position **r** and moving with linear momentum **p** as in Figure 11.3. In describing linear motion, we found that the net force on the particle equals the time rate of change of its linear momentum, $\sum\mathbf{F} = d\mathbf{p}/dt$ (see Eq. 9.3). Let us take the cross product of each side of Equation 9.3 with **r**, which gives us the net torque on the particle on the left side of the equation:

$$\mathbf{r}\times\sum\mathbf{F} = \sum\boldsymbol{\tau} = \mathbf{r}\times d\mathbf{p}/dt$$

Now let us add to the right-hand side the term $d\mathbf{r}/dt\times\mathbf{p}$, which is zero because $d\mathbf{r}/dt = \mathbf{v}$ and **v** and **p** are parallel. Thus,

$$\sum\boldsymbol{\tau} = \mathbf{r}\times d\mathbf{p}/dt + d\mathbf{r}/dt\times\mathbf{p}$$

We recognize the right-hand side of this equation as the derivative of $\mathbf{r}\times\mathbf{p}$ (see Equation
11.6). Therefore,

$$\sum\boldsymbol{\tau} = d(\mathbf{r}\times\mathbf{P})/dt \qquad (11.9)$$

This looks very similar in form to Equation 9.3, $\sum \mathbf{F} = d\mathbf{p}/dt$. This suggests that the combination $\mathbf{r} \times \mathbf{p}$ should play the same role in rotational motion that $\mathbf{p}$ plays in translational motion. We call this combination the *angular momentum* of the particle:

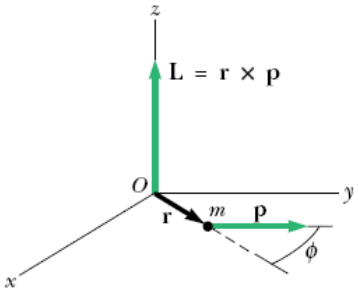


Figure 11.3 The angular momentum **L** of a particle of mass *m* and linear momentum **p** located at the vector position **r** is a vector given by $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. The value of **L** depends on the origin about which it is measured and is a vector perpendicular to both **r** and **p**.

The instantaneous **angular momentum L** of a particle relative to the origin *O* is defined by the cross product of the particle's instantaneous position vector **r** and its instantaneous linear momentum **p**:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad \textbf{Angular momentum of a particle} \qquad (11.10)$$

This allows us to write Equation 11.9 as

$$\sum \boldsymbol{\tau} = d\mathbf{L}/dt \qquad (11.11)$$

Which is the rotational analog of Newton's second law, $\mathbf{F} = d\mathbf{p}/dt$. Equation 11.11 states that the torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

Note that Equation 11.11 is valid only if $\sum \boldsymbol{\tau}$ and **L** are measured about the same origin. The SI unit of angular momentum is $kg.m^2/s$. Following the right-hand rule, we see that the direction of **L** is perpendicular to the plane formed by **r** and **p**. In Figure 11.3, **r** and **p** are in the *xy* plane, and so **L** points in the *z* direction. Because $\mathbf{p} = m\mathbf{v}$, the magnitude of **L** is

$$L = mvr \sin \phi \qquad (11.12)$$

Where, ϕ is the angle between **r** and **p**. It follows that *L* is zero when **r** is parallel to **p** ($\phi = 0$ or 180°). In other words, when the linear velocity of the particle is along a line that passes through the origin, the particle has zero angular momentum with respect to the origin. On the other hand, if **r** is perpendicular to **p** ($\phi = 90°$), then $L = mvr$. At that instant, the particle moves exactly as if it were on the rim of a wheel rotating about the origin in a plane defined by **r** and **p**.

Angular Momentum of a System of Particles

In Section 9.6, we showed that Newton's second law for a particle could be extended to
a system of particles, resulting in:

$$\sum \mathbf{F}_{ext} = d\mathbf{p}_{tot}/dt$$

The total angular momentum of a system of particles about some point is defined as the vector sum of the angular momenta of the individual particles:

$$\mathbf{L}_{tot} = \mathbf{L}_1 + \mathbf{L}_2 + \ldots + \mathbf{L}_n = \sum \mathbf{L}_i$$

Where the vector sum is over all *n* particles in the system.
Let us differentiate this equation with respect to time:

$$d\mathbf{L}_{tot}/dt = \sum d\mathbf{L}_i/dt = \sum \boldsymbol{\tau}_i$$

Where we have used Equation 11.11 to replace the time rate of change of the angular momentum of each particle with the net torque on the particle.

We conclude that the total angular momentum of a system can vary with time only if a net external torque is acting on the system, so that we have

$$\sum \boldsymbol{\tau}_{ext} = d\mathbf{L}_{tot}/dt \qquad (11.13)$$

That is, the net external torque acting on a system about some axis passing through an origin in an inertial frame equals the time rate of change of the total angular momentum of the system about that origin.

Although we do not prove it here, the following statement is an important theorem concerning the angular momentum of a system relative to the system's center of mass: *The resultant torque acting on a system about an axis through the center of mass equals the time rate of change of angular momentum of the system regardless of the motion of the center of mass.*
This theorem applies even if the center of mass is accelerating, provided $\boldsymbol{\tau}$ and $\mathbf{L}$ are evaluated relative to the center of mass.

11.3 Angular Momentum of a Rotating Rigid Object

Consider a nondeformable—a rigid object rotating about a fixed axis that coincides with the z axis of a coordinate system, as shown in Figure 11.4.

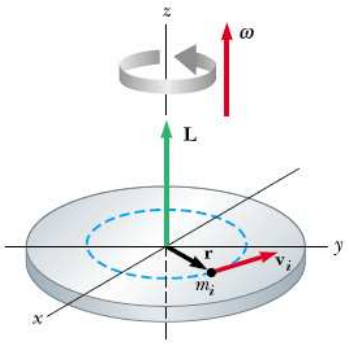


Figure 11.4 When a rigid object rotates about an axis, the angular momentum $\mathbf{L}$ is in the same direction as the angular velocity ω, according to the expression $\mathbf{L} = I\omega$.

Let us determine the angular momentum of this object. Each *particle* of the object rotates in the xy plane about the z axis with an angular speed ω. The magnitude of the angular momentum of a particle of mass m_i about the z axis is $m_i v_i r_i$. Because $v_i = r_i\omega$, we can express the magnitude of the angular momentum of this particle as

$$L_i = m_i r_i^2 \omega$$

The vector $\mathbf{L}_i$ is directed along the z axis, as is the vector ω.
We can now find the angular momentum (which in this situation has only a z component) of the whole object by taking the sum of L_i over all particles:

$$L_z = \sum L_i = \sum m_i r_i^2 \omega = (\sum m_i r_i^2)\,\omega$$

$$L_z = I\,\omega \qquad (11.4)$$

Where we have recognized $\sum m_i r_i^2$ as the moment of inertia I of the object about the z axis (Equation 10.15).
Now let us differentiate Equation 11.14 with respect to time, noting that I is constant for a rigid object:

$$dL_z/dt = I\,d\omega/dt = I\alpha \qquad (11.15)$$

Where α is the angular acceleration relative to the axis of rotation. Because dL_z/dt is equal to the net external torque (see Eq. 11.13), we can express Equation 11.15 as

$$\sum \tau_{ext} = I\alpha \qquad \textbf{Rotational form of Newton's second law} \qquad (11.16)$$

That is, the net external torque acting on a rigid object rotating about a fixed axis equals the moment of inertia about the rotation axis multiplied by the object's angular acceleration relative to that axis. This result is the same as Equation 10.21, which was derived using a force approach, but we derived Equation 11.16 using the concept of angular momentum. This equation is also valid for a rigid object rotating about a moving axis provided the moving axis (1) passes through the center of mass and (2) is a symmetry axis.
If a symmetrical object rotates about a fixed axis passing through its center of mass, you can write Equation 11.14 in vector form as $\mathbf{L} = I\omega$, where $\mathbf{L}$ is the total angular momentum of the object measured with respect to the axis of rotation. Furthermore, the expression is valid for any object, regardless of its symmetry, if $\mathbf{L}$ stands for the component of angular momentum along the axis of rotation.

11.4 Conservation of Angular Momentum

In Chapter 9 we found that the total linear momentum of a system of particles remains constant if the system is isolated, that is, if the resultant external force acting on the system is zero. We have an analogous conservation law in rotational motion:

The conservation of angular momentum: is defined as, the total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero, that is, if the system is isolated.

This follows directly from Equation 11.13, which indicates that if

$$\sum \boldsymbol{\tau}_{\mathbf{ext}} = d\mathbf{L}_{\mathbf{tot}}/dt = 0 \qquad (11.7)$$

Then

$$\mathbf{L}_{\text{tot}} = \text{constant} \qquad \text{or} \qquad \mathbf{L}_i = \mathbf{L}_f \qquad (11.8)$$

For an isolated system consisting of a number of particles, we write this conservation law as $\mathbf{L}_{\text{tot}} = \sum \mathbf{L}_n = \text{constant}$, where the index n denotes the nth particle in the system.

If the mass of an isolated rotating system undergoes redistribution in some way, the system's moment of inertia changes. Because the magnitude of the angular momentum of the system is $L = I\,\omega$ (Eq. 11.14), conservation of angular momentum requires that the product of I and ω must remain constant. Thus, a change in I for an isolated system requires a change in ω. In this case, we can express the principle of conservation of angular momentum as

$$I_i\,\omega_i = I_f\,\omega_f = \text{constant} \qquad (11.19)$$

This expression is valid both for rotation about a fixed axis and for rotation about an axis through the center of mass of a moving system as long as that axis remains fixed in direction.

In Equation 11.18 we have a third conservation law to add to our list. We can now state that the energy, linear momentum, and angular momentum of an isolated system all remain constant:

$E_i = E_f$ $\quad \mathbf{p}_i = \mathbf{p}_f$ $\quad \mathbf{L}_i = \mathbf{L}_f$ $\quad$ For an isolated system

11.5 The Motion of Gyroscopes and Tops

A very unusual and fascinating type of motion you probably have observed is that of a top spinning about its axis of symmetry, as shown in Figure 11.5a. If the top spins very rapidly, the symmetry axis rotates about the z axis, sweeping out a cone (see Fig. 11.5b). The motion of the symmetry axis about the vertical—known as **precessional motion**—is usually slow relative to the spinning motion of the top.

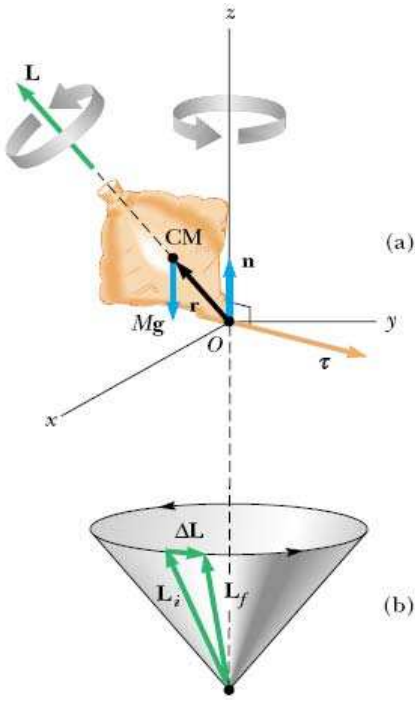


Figure 11.5 Precessional motion of a top spinning about its symmetry axis. (a) The only external forces acting on the top are the normal force **n** and the gravitational force $M\mathbf{g}$. The direction of the angular momentum **L** is along the axis of symmetry. The right hand rule indicates that $\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times M\mathbf{g}$ is in the xy plane. (b) The direction of $\Delta\mathbf{L}$ is parallel to that of τ in part (a). The fact that $\mathbf{L}_f = \Delta\mathbf{L} + \mathbf{L}_i$ indicates that the top precesses about the z axis.

It is quite natural to wonder why the top does not fall over. Because the center of mass is not directly above the pivot point O, a net torque is clearly acting on the top about O—a torque resulting from the gravitational force $M\mathbf{g}$. The top would certainly fall over if it were not spinning. Because it is spinning, however, it has an angular momentum **L** directed along its symmetry axis.

The essential features of precessional motion can be illustrated by considering the simple gyroscope shown in Figure 11.6a. The two forces acting on the top are the downward gravitational force $M\mathbf{g}$ and the normal force n acting upward at the pivot point O. The normal force produces no torque about the pivot because its moment arm through that point is zero. However, the gravitational force produces a torque $\tau = \mathbf{r} \times M\mathbf{g}$ about O, where the direction of τ is perpendicular to the plane formed by **r** and $M\mathbf{g}$. The net torque and angular momentum of the gyroscope are related through Equation 11.13:

$$\tau = d\mathbf{L}/dt$$

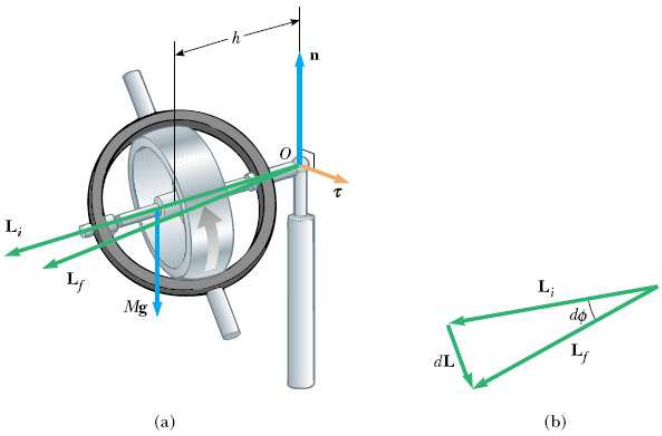


Figure 11.6 (a) The motion of a simple gyroscope pivoted a distance h from its center of mass. The gravitational force $M\mathbf{g}$ produces a torque about the pivot, and this torque is perpendicular to the axle. (b) Overhead view of the initial and final angular momentum vectors. The torque results in a change in angular momentum $d\mathbf{L}$ in a direction perpendicular to the axle. The axle sweeps out an angle $d\phi$ in a time interval dt.

Figure 11.6b illustrates the resulting precessional motion of the symmetry axis of the gyroscope. In a time interval dt, the change in angular momentum is $d\mathbf{L} = \mathbf{L}_f - \mathrm{L}_i = \tau\, dt$.
To simplify the description of the system, we shall neglect the contribution from the center-of-mass motion and take the total angular momentum to be just $I\omega$.
From the vector triangle formed by the vectors $\mathbf{L}_i$, $\mathbf{L}_f$, and $d\mathbf{L}$, we see that

$$\sin(d\phi) \approx d\phi = \frac{dL}{L} = \frac{\tau dt}{L} = \frac{(Mgh)dt}{L}$$

Where we have used the fact that, for small values of any angle θ, sin θ ≈ θ. Dividing through by dt and using the relationship $L = I\omega$, we find that the rate at which the axle rotates about the vertical axis is

$$\omega_p = \frac{d\phi}{dt} = \frac{Mgh}{I\omega} \qquad (11.20)$$

The angular speed ω_p is called the **precessional** frequency. This result is valid only when $\omega_p << \omega$. Otherwise, a much more complicated motion is involved. As you can see from Equation 11.20, the condition $\omega_p << \omega$ is met when ω is large, that is, the wheel spins rapidly. Furthermore, note that the precessional frequency decreases as ω increases—that is, as the wheel spins faster about its axis of symmetry.

11.6 Angular Momentum as a Fundamental Quantity

The concept of angular momentum is valid on a submicroscopic scale and has been used extensively in the development of modern theories of atomic, molecular, and nuclear physics. In these developments, it has been found that the angular momentum of a system is a fundamental quantity. The word *fundamental* in this context implies that angular momentum is an intrinsic property of atoms, molecules, and their constituents, a property that is a part of their very nature.

To explain the results of a variety of experiments on atomic and molecular systems, we rely on the fact that the angular momentum has discrete values. These discrete values are multiples of the fundamental unit of angular momentum $\hbar = h/2\pi$, where h is called Planck's constant:
Fundamental unit of angular momentum = $\hbar = 1.054 \times 10^{-34}$ kg.m^2/s.
To estimate the angular speed of a diatomic molecule, let us consider the O_2 molecule as a rigid rotor, that is, two atoms separated by a fixed distance d and rotating about the center of mass (Fig. 11.7). Equating the angular momentum to the fundamental unit, we can find the order of magnitude of the lowest angular speed:

$$\omega \approx \frac{\hbar}{I_{CM}} \quad \text{or} \quad I_{CM}\omega \approx \hbar$$

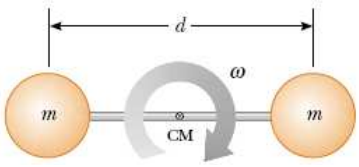


Figure 11.7 The rigid-rotor model of a diatomic molecule. The rotation occurs about the center of mass in the plane of the page.

It was found that the moment of inertia of the O_2 molecule about this axis of rotation is 1.95×10^{-46} kg.m^2. Therefore,

$$\omega \approx \frac{\hbar}{I_{CM}} = \frac{1.054 \times 10^{-34}\, kg.m^2/s}{1.95 \times 10^{-46}\, kg.m^2} \approx 10^{12}\, rad/s$$

Actual angular speeds are found to be multiples of a number with this order of magnitude.

This simple example shows that certain classical concepts and models, when properly modified, are useful in describing some features of atomic and molecular systems. A wide variety of phenomena on the submicroscopic scale can be explained only if we assume discrete values of the angular momentum associated with a particular type of motion.

11.7 Homework

The Vector Product and Torque

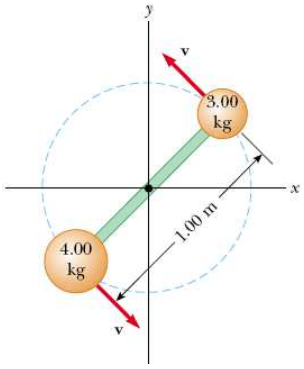
11.7.1 Given **M**= 6*i* + 2*j* - *k* and **N**=2*i* - *j* – 3*k*, calculate the vector product **M**×**N**.

11.7.2 Two vectors are given by **A** = 3**i** + 4**j** and **B** = 2**i** + 3**j**. Find (a) **A**×**B** and (b) the angle between **A** and **B**.

11.7.3 A particle is located at the vector position **r** = (**i** + 3**j**) m, and the force acting on it is **F** = (3**i** + 2**j**) N. What is the torque about (a) the origin and (b) the point having coordinates (0, 6) m?

Angular Momentum

11.7.4 A light rigid rod 1 m in length joins two particles, with masses 4 kg and 3 kg, at its ends. The combination rotates in the *xy* plane about a pivot through the center of the rod (see the opposite Figure). Determine the angular momentum of the system about the origin when the speed of each particle is 5 m/s.



From equation (11.12), the angular momentum: $L = mvr \sin \phi$

Where, ϕ is the angle between **r** and **p**. But since **r** is perpendicular to **p** ($\phi = 90°$), then $L = mvr$.

$L = \sum m_i v_i r_i$

= (4 kg)(5m/s)(0.5 m) + (3 kg)(5 m/s)(0.5 m)

L = (17.5 kg.m^2/s), and

L = 17.5 kg.m^2/s) **k**

11.7.5 The position vector of a particle of mass 2 kg is given as a function of time by **r** = (6**i** + 5 *t* **j**) m. Determine the angular momentum of the particle about the origin, as a function of time.

r = (6**i** + 5*t* **j**) m **v** = d**r**/dt = 5 **j** (m/s)

p = *m***v** = 2 kg [5 **j** (m/s)] = 10 **j** (kg.m^2/s)

Angular Momentum of a Rotating Rigid Object

11.7.7 A uniform solid disk of mass 3 kg and radius 0.2 m rotates about a fixed axis perpendicular to its face. If the angular frequency of rotation is 6 rad/s, calculate the

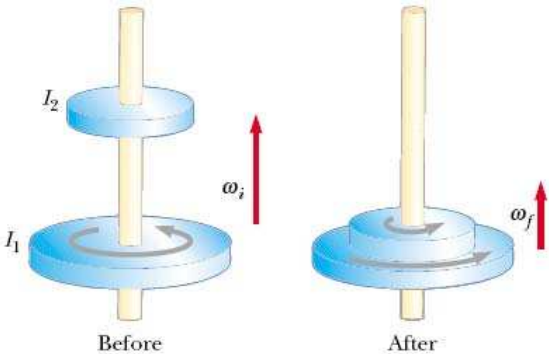
angular momentum of the disk when the axis of rotation (a) passes through its center of mass and (b) passes through a point midway between the center and the rim.

(a) By using table 10.2, $I = ½ MR^2$
$L = I\omega = (½ MR^2)\,\omega = ½$ (3 kg) (0.2 m)^2 (6 rad/s) = 0.36 kg.m^2/s

(b) $L = ½ MR_1^2 + MR_2^2) = ½ MR^2 + M(R/2)^2 = ½ MR^2 + M R^2/4 = ¾ MR^2\omega$
$L = ¾$ (3 kg) (0.2 m)^2 (6 rad/s) = 0.54 kg.m^2/s

Conservation of Angular Momentum

11.7.8 A cylinder with moment of inertia I_1 rotates about a vertical, frictionless axle with angular speed ω_i. A second cylinder, this one having moment of inertia I_2 and initially not rotating, drops onto the first cylinder (Opposite Figure). Because of friction between the surfaces, the two eventually reach the same angular speed ω_f. (a) Calculate ω_f. (b) Show that the kinetic energy of the system decreases in this interaction, and calculate the ratio of the final to the initial rotational energy.



The Motion of Gyroscopes and Tops

11.7.9 (new/old, P 11.40/38, solved) A spacecraft is in empty space. It carries on board a gyroscope with a moment of inertia of I_g = 20 kg.m^2 about the axis of the gyroscope. The moment of inertia of the spacecraft around the same axis is $I_s = 5 \times 10^5$ kg.m^2. Neither the spacecraft nor the gyroscope is originally rotating. The gyroscope can be powered up in a negligible period of time to an angular speed of 100 s^{-1}. If the orientation of the spacecraft is to be changed by 30°, for how long should the gyroscope be operated?

Angular momentum of the system and spacecraft and the gyroscope is conserved. The gyroscope and spacecraft turn in the opposite directions.

$$0 = I_1\omega_1 + I_2\omega_2 \qquad -I_1\omega_1 = I_2\omega_2 = I_2\frac{\theta}{t}$$

$$-20kg.m^2(-100rad/s) = 5\times10^5 kg.m^2\left(\frac{30^o}{t}\right)\left(\frac{\pi rad}{180^o}\right) \qquad t = \frac{2.62\times10^5 s}{2000} = 131s$$

Angular Momentum as a Fundamental Quantity

11.7.10 In the Bohr model of the hydrogen atom, the electron moves in a circular orbit of radius 0.529×10^{-10} m around the proton. Assuming the orbital angular momentum of the electron is equal to $h/2\pi$, calculate (a) the orbital speed of the electron, (b) the kinetic energy of the electron, and (c) the angular frequency of the electron's motion.

Chapter 12

Static Equilibrium and Elasticity

The term *equilibrium* implies either that the object is at rest or that its center of mass moves with constant velocity relative to the observer. We deal her only with the former case, in which the object is in *static equilibrium*. Static equilibrium represents a common situation in engineering practice, and the principles it involves are of special interest to civil engineers, architects, and mechanical engineers.

12.1 The Conditions for Equilibrium

If the object can be modeled as a particle, then the condition for static equilibrium is that the net force acting on an object must be zero. For the extended object which can not be modeled as a particle then the condition for static equilibrium involves the net torque acting on it.

In Figure 12.1 the torque associated with the force **F** about *O*, is given by Equation 11.1 (refer to chapter 11):

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

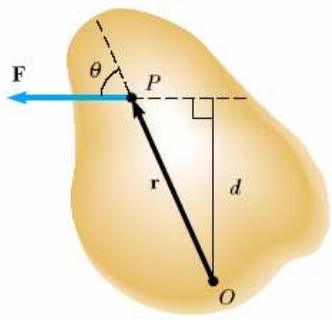


Figure 12.1 A single *force* **F** acts on a rigid object at the point *P*.

The tendency of **F** to rotate the object about an axis through *O* depends on the moment arm *d*, as well as on the magnitude of **F**.

There are two necessary conditions for equilibrium of an object:

1. The resultant external force must equal zero:

$$\sum \mathbf{F} = 0 \qquad (12.1)$$

2. The resultant external torque about *any* axis must be zero:

$$\sum \boldsymbol{\tau} = 0 \qquad (12.2)$$

We restrict our discussion to situations in which all the forces lie in the *xy* plane. With this restriction, we must deal with only three scalar equations. Two of these come from balancing the forces in the *x* and *y* directions. The third is that the net torque about a perpendicular axis through *any* point in the *xy* plane must be zero. Hence, the conditions of equilibrium provide the equations

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum \tau_z = 0 \qquad (12.3)$$

Consider an object being acted on by several forces such that the resultant force

$$\sum \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots = 0$$

Figure 12.2 describes this situation. The point of application of $\mathbf{F}_1$ relative to O is specified by the position vector $\mathbf{r}_1$. Similarly, the points of application of $\mathbf{F}_2$, $\mathbf{F}_3$, . . . are specified by $\mathbf{r}_2$, $\mathbf{r}_3$, . . . (not shown). The net torque about an axis through O is

$$\sum \boldsymbol{\tau}_O = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \ldots$$

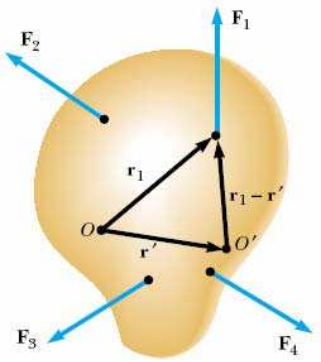


Figure 12.2 Construction showing that if the net torque is zero about origin O, it is also zero about any other origin, such that O'.

Now consider another arbitrary point O' having a position vector $\mathbf{r}'$ relative to O. The point of application of $\mathbf{F}_1$ relative to O' is identified by the vector $\mathbf{r}_1 - \mathbf{r}'$. Like-wise, the point of application of $\mathbf{F}_2$ relative to O' is $\mathbf{r}_2 - \mathbf{r}'$, and so forth. Therefore, the torque about an axis through O' is

$$\begin{aligned}\sum \boldsymbol{\tau}_{O'} &= (\mathbf{r}_1 - \mathbf{r}') \times \mathbf{F}_1 + (\mathbf{r}_2 - \mathbf{r}') \times \mathbf{F}_2 + (\mathbf{r}_3 - \mathbf{r}') \times \mathbf{F}_3 + \ldots \\ &= \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 + \mathbf{r}_3 \times \mathbf{F}_3 + \ldots - \mathbf{r}' \times (\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots)\end{aligned}$$

Because the net force is assumed to be zero (given that the object is in translational equilibrium), the last term vanishes (because $\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$ = zero), and we see that the torque about an axis through O' is equal to the torque about an axis through O (i.e. $\sum \boldsymbol{\tau}_O = \sum \boldsymbol{\tau}_{O'}$). Hence, **if an object is in translational equilibrium and the net torque is zero about one axis, then the net torque must be zero about any other axis.**

12.2 More on the Center of Gravity

Two equal-magnitude but oppositely directed forces result in equilibrium if they are applied at the same point on an object. However, if the point of application of one of the forces is moved, so that the two forces no longer act along the same line of action, then the object undergoes an angular acceleration.

Whenever we deal with a rigid object, one of the forces we must consider is the gravitational force acting on it, and we must know the point of application of this force.

To compute the torque due to the gravitational force on an object of mass M, we need only consider the force $M\mathbf{g}$ acting at the center of gravity of the object. How do we find this special point? If we assume that $\mathbf{g}$ is uniform over the object, then the center of gravity of the object coincides with its center of mass. Consider an object of arbitrary shape lying in the xy plane, as illustrated in Figure 12.3. Suppose the object is divided into a large number of particles of masses m_1, m_2, m_3, . . . having coordinates (x_1, y_1), (x_2, y_2), (x_3, y_3), ... In Equation 9.28 we defined the x coordinate of the center of mass of such an object to be

$$x_{CM} = \frac{m_1 x_1 + x_2 m_2 + x_3 m_3 + ... + m_n x_n}{m_1 + m_2 + m_3 + ... + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

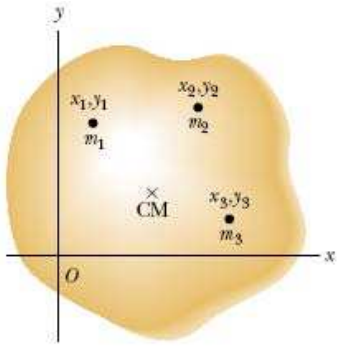


Figure 12.3 An object can be divided into many small particles each having a specific mass and specific coordinates. These particles can be used to locate the center of mass.

We use a similar equation to define the y coordinate of the center of mass, replacing each x with its y counterpart. Let us now examine the situation from another point of view by considering the gravitational force exerted on each particle, as shown in Figure 12.4.

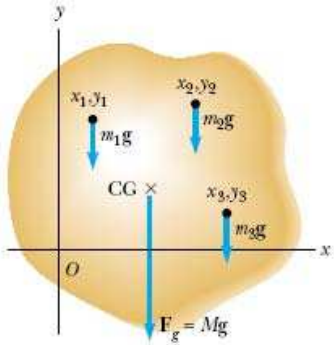


Figure 12.4 The center of gravity of an object is located at the center of mass if **g** is constant over the object.

The magnitude of the torque due to the force $m_1\mathbf{g}_1$ is $m_1 g_1 x_1$, where g_1 is the value of the gravitational acceleration at the position of the particle of mass m_1. We wish to locate the center of gravity, the point at which application of the single gravitational force Mg. Equating the torque resulting from Mg acting at the center of gravity to the sum of the torques acting on the individual particles gives

$$(m_1 g_1 + m_2 g_2 + m_3 g_3 + ...) x_{CG} = m_1 g_1 x_1 + m_2 g_2 x_2 + m_3 g_3 x_3 + ...$$

If we assume uniform g over the object, then the g terms cancel and we obtain

$$x_{CG} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + ...}{m_1 + m_2 + m_3 + ...} \qquad (12.4)$$

Comparing this result with Equation 9.28, we see that **the center of gravity is located at the center of mass as long as g is uniform over the entire object**.

12.3 Examples of Rigid Objects in Static Equilibrium

The photograph of the one-bottle holder in Figure 12.5 shows one example of a balanced mechanical system that seems to defy gravity. For the system (holder plus bottle) to be in equilibrium, the net external force must be zero (see Eq. 12.1) and the net external torque must be zero (see Eq. 12.2). The second condition can be satisfied only when the center of gravity of the system is directly over the support point.
When working static equilibrium problems, you must recognize all the external forces acting on the object. Failure to do so results in an incorrect analysis.

Figure 12.5 This is one-bottle holder is a surprising display of static equilibrium. The center of gravity of the system (bottle plus holder) is directly over the support point.

Example 12.1 (solved) The Seesaw Revisited

A seesaw consisting of a uniform board of mass M and length l supports a father and daughter with masses m_f and m_d, respectively, as shown in Figure 12.6. The support (called the *fulcrum*) is under the center of gravity of the board, the father is a distance d from the center, and the daughter is a distance $l/2$ from the center. (A) Determine the magnitude of the upward force **n** exerted by the support on the board. (B) Determine where the father should sit to balance the system.

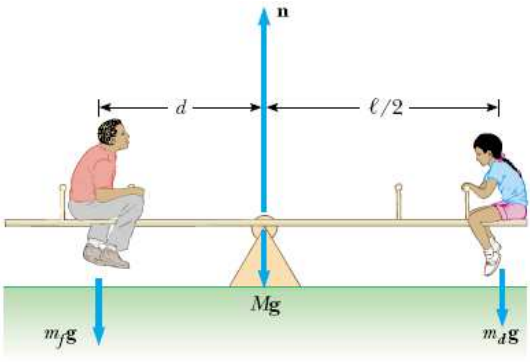


Figure 12.6 A balanced system.

Solution:

(A) The upward force **n** must balance all the downward forces. From $\sum F_y = 0$, and defining upward as the positive y direction, we have

$$n - m_f g - m_d g - Mg = 0$$

$$n = m_f g + m_d g + Mg$$

(B) The torques produced by **n** and the gravitational force acting on the board are zero. We see from $\Sigma\tau = 0$ that

$$(m_g g)(d) - (m_d g)(\frac{l}{2}) = 0$$

$$d = (\frac{m_d}{m_f})\frac{1}{2}l$$

Example 12.2 A Weighted Hand

A person holds a 50 N sphere in his hand. The forearm is horizontal, as shown in Figure 12.7a. The biceps muscle is attached 3 cm from the joint, and the sphere is 35 cm from the joint. Find the upward force exerted by the biceps on the forearm and the downward force exerted by the upper arm on the forearm and acting at the joint. Neglect the weight of the forearm.

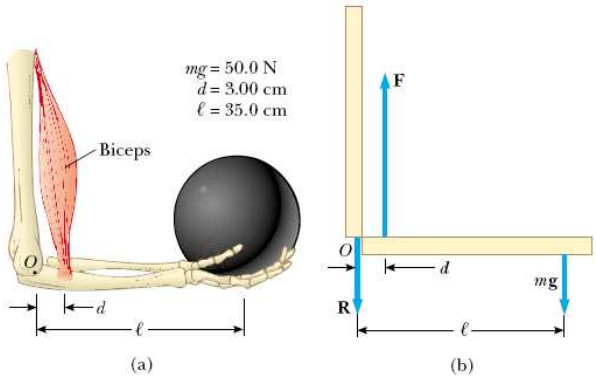


Figure 12.7 (Example 12.2) (a) The biceps muscle pulls upward with a force F that is essentially at a right angle to the forearm. (b) The mechanical model for the system described in part (a).

Solution: As shown in Figure 12.7b, **F** is the upward force exerted by the biceps and **R** is the downward force exerted by the upper arm at the joint. From the first condition for equilibrium, we have, with upward as the positive y direction,

$$\sum F_y = F - R - 50\ \text{N} = 0 \qquad (1)$$

From the second condition for equilibrium, we know that the sum of the torques about any point must be zero. With the joint O as the axis, we have

$$\sum \tau = Fd - mgl = 0$$

$$F\,(3\ \text{cm}) - (50\ \text{N})(35\ \text{cm}) = 0$$

$$F = 583\ \text{N}$$

This value for F can be substituted into Equation (1) to give

$$R = 533\ \text{N}$$

As this example shows, the forces at joints and in muscles can be extremely large.

Example 12.3 (solved) The Leaning Ladder

A uniform ladder of length l rests against a smooth, vertical wall (Fig. 12.8a). If the mass of the ladder is m and the coefficient of static friction between the ladder and the ground is $\mu_s = 0.4$, find the minimum angle θ_{min} at which the ladder does not slip.

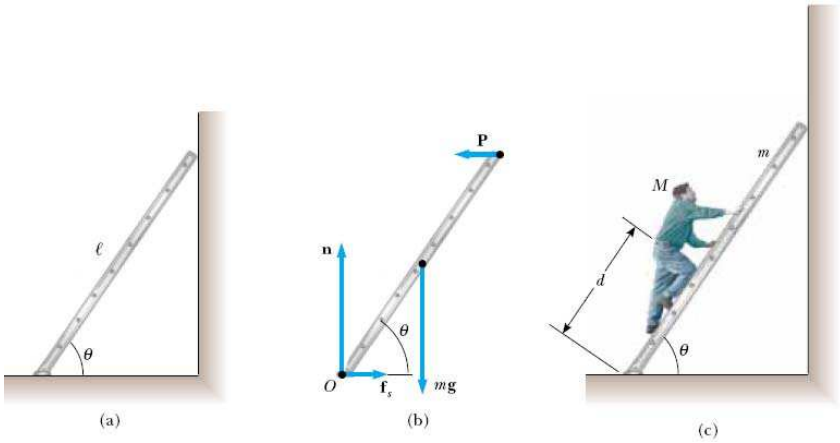


Figure 12.8 (Example 12.3) (a) A uniform ladder at rest, leaning against a smooth wall. The ground is rough. (b) The free-body diagram for the ladder. (c) A person of mass M begins to climb the ladder when it is at the minimum angle found in part (a) of the example. Will the ladder slip?

Solution: The free-body diagram showing all the external forces acting on the ladder is illustrated in Figure 12.8b. The force exerted by the ground on the ladder is the vector sum of a normal force **n** and the force of static friction $\mathbf{f}_s$. The reaction force **P** exerted by the wall on the ladder is horizontal because the wall is frictionless. Applying the first condition for equilibrium to the ladder, we have

$$\sum F_x = f_s - p = 0 \qquad (1)$$

$$\sum F_y = n - mg = 0 \qquad (2)$$

The first equation tells us that $P = f_s$. From the second equation we see that $n = mg$. Furthermore, when the ladder is on the verge of slipping, the force of friction must be a maximum, which is given by

$$f_{s,\,\text{max}} = \mu_s n$$

(Recall Eq. 5.8: $f_s \leq \mu_s n$.) Thus, we must have

$$P = f_s = \mu_s n = \mu_s mg$$

To find θ_{min}, we must use the second condition for equilibrium. When we take the torques about an axis through the origin O at the bottom of the ladder, we have

$$\sum \tau_O = Pl\sin\theta - mg\frac{l}{2}\cos\theta = 0 \qquad (3)$$

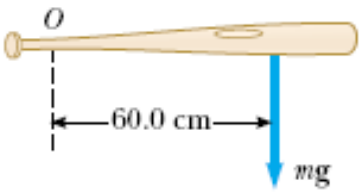
This expression gives

$$\tan\theta_{\min} = \frac{mg}{2P} = \frac{mg}{2\mu_s mg} = \frac{1}{2\mu_s} = 1.25 \qquad \theta_{\min} = 51^o$$

12.4 Homework

The Conditions for Equilibrium of a Rigid Body

12.4.1 A baseball player holds a 36-oz bat (weight = 10 N) with one hand at the point *O* (opposite figure). The bat is in equilibrium. The weight of the bat acts along a line 60 cm to the right of *O*. Determine the force and the torque exerted by the player on the bat around an axis through *O*.

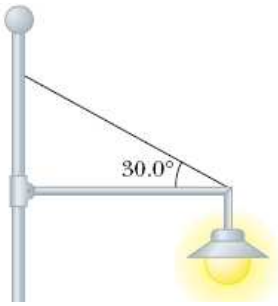


More on the Center of Gravity

12.4.2 (P12.7/5 old/new, solved) Consider the following mass distribution: 5kg at (0, 0) m, 3 kg at (0, 4) m, and 4 kg at (3, 0) m. Where should a fourth object of mass 8 kg be placed so that the center of gravity of the four-object arrangement will be at (0, 0)?

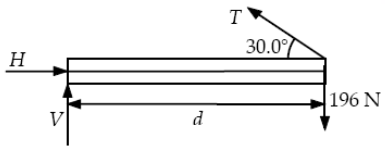
Examples of Rigid Objects in Static Equilibrium

12.4.2 A 20 kg floodlight in a park is supported at the end of a horizontal beam of negligible mass that is hinged to a pole, as shown in the opposite Figure. A cable at an angle of 30° with the beam helps to support the light. Find (a) the tension in the cable and (b) the horizontal and vertical forces exerted on the beam by the pole.



Solution:

(a) Consider the torque about axis perpendicular to the page and through the left end of the horizontal beam.



$$\sum \tau = F_y d - mgd = 0$$
$$(T\sin 30^o)d - (20\text{ kg})(9.8\text{ m/s}^2)d = 0$$
$$0.5\,T = 196\text{ N}$$
$$T = 392\text{ N}$$

(b) From $\sum F_x = 0$, $H - T\cos 30^o = 0$, or $H = (392\text{N})\,(0.866) = 339.5\text{N}$ to the right
From $\sum F_y = 0$, $V + T\sin 30^o - \text{mg} = 0$, or $V = (196\text{N}) - (392\text{N})\,(0.5) = 0$

Chapter 13

Universal Gravitation

Before 1687, a large amount of data had been collected on the motions of the Moon and the planets, but a clear understanding of the forces related to these motions was not available. In that year, Isaac Newton provided the key that unlocked the secrets of the heavens. He knew, from his first law, that a net force had to be acting on the Moon because without such a force the Moon would move in a straight-line path rather than in its almost circular orbit. Newton reasoned that this force was the gravitational attraction exerted by the Earth on the Moon.

13.1 Newton's Law of Universal Gravitation

You may have heard the legend that Newton was struck on the head by a falling apple while napping under a tree. This alleged accident supposedly prompted him to imagine that perhaps all objects in the Universe were attracted to each other in the same way the apple was attracted to the Earth. Newton analyzed astronomical data on the motion of the Moon around the Earth. From that analysis, he made the bold assertion that the force law governing the motion of planets was the *same* as the force law that attracted a falling apple to the Earth.

In 1687 Newton published his work on the law of gravity in his treatise *Mathematical Principles of Natural Philosophy.* **Newton's law of universal gravitation** states that ***every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them***.

If the particles have masses $m1$ and $m2$ and are separated by a distance r, the magnitude of this gravitational force is

$$F_g = G\frac{m_1 m_2}{r^2} \qquad (13.1)$$

Where G is a constant, called the *universal gravitational constant,* that has been measured experimentally. Its value in SI units is

$$G = 6.673 \times 10^{-11}\ \text{N.m}^2/\text{kg}^2 \qquad (13.2)$$

Henry Cavendish (1731–1810) measured the universal gravitational constant in an important 1798 experiment. Cavendish's apparatus consists of two small spheres, each of mass m, fixed to the ends of a light, horizontal rod suspended by a fine fiber or thin metal wire, as illustrated in Figure 13.1. When two large spheres, each of mass M, are placed near the smaller ones, the attractive force between smaller and larger spheres causes the rod to rotate and twist the wire suspension to a new equilibrium orientation. The angle of rotation is measured by the deflection of a light beam reflected from a mirror attached to the vertical suspension.

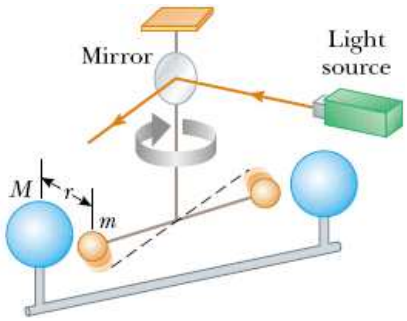


Figure 13.1 Cavendish apparatus for measuring G. The dashed line represents the original position of the rod.

The form of the force law given by Equation 13.1 is often referred to as an **inverse-square law** because the magnitude of the force varies as the inverse square of the separation of the particles. We can express this force in vector form by defining a unit vector $\vec{r}_{12}$ (Fig. 13.2). Because this unit vector is directed from particle 1 toward particle 2, the force exerted by particle 1 on particle 2 is

$$\vec{F}_{12} = -G\frac{m_1 m_2}{r^2}\vec{r}_{12} \qquad (13.3)$$

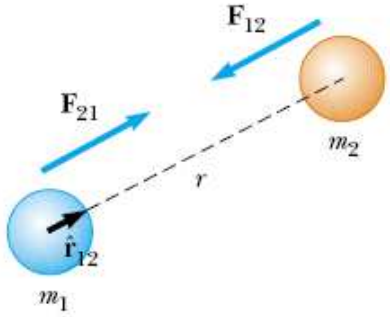


Figure 13.2 The gravitational force between two particles is attractive. The unit vector $\vec{r}$ is directed from particle 1 toward particle 2. Note that **F**21= -**F**12.

Where the negative sign indicates that particle 2 is attracted to particle 1, and hence the force on particle 2 must be directed toward particle 1. By Newton's third law, the force exerted by particle 2 on particle 1, designated **F**21, is equal in magnitude to **F**12 and in the opposite direction. That is, these forces form an action–reaction pair, and **F**21 = - **F**12.

Two features of Equation 13.3 deserve mention. First, the gravitational force is a field force that always exists between two particles, regardless of the medium that separates them. Because the force varies as the inverse square of the distance between the particles, it decreases rapidly with increasing separation.

Equation 13.3 can also be used to show that **the gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass as if the distribution were concentrated at the center.** For example, the magnitude of the force exerted by the Earth on the particle of mass *m* near the Earth's surface is

$$F_g = G\frac{M_E m}{R_E^2} \qquad (13.4)$$

Where *ME* is the Earth's mass and *RE* its radius. This force is directed toward the center of the Earth.

Example 13.1 Billiards, Anyone? (Discussed)

Three 0.3 kg billiard balls are placed on a table at the corners of a right triangle, as shown in Figure 13.3. Calculate the gravitational force vector on the cue ball (designated *m*1) resulting from the other two balls.

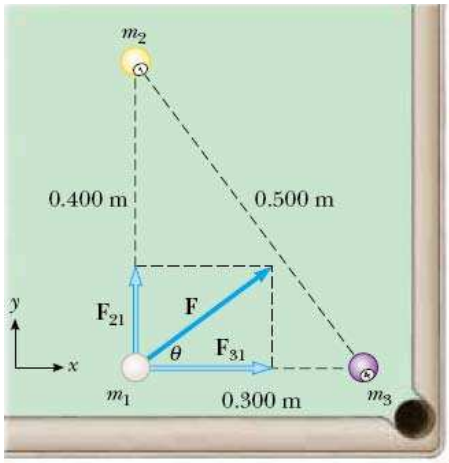


Figure 13.3 The resultant gravitational force acting on the cue ball is the vector sum **F**21 & **F**31.

Solution

First we calculate separately the individual forces on the cue ball due to the other two balls, and then we find the vector sum to obtain the resultant force. We can see graphically that this force should point upward and toward the right. We locate our coordinate axes as shown in Figure 13.3, placing our origin at the position of the cue ball.
The force exerted by *m*2 on the cue ball is directed upward and is given by

$$\vec{F}_{21} = G\frac{m_2 m_1}{r_{21}^2}\vec{j} = (6.67\times10^{-11}\,N.m^2/kg^2)\frac{(0.3kg)(0.3kg)}{(0.4m)^2} = 3.75\times10^{-11}\vec{j}N$$

This result shows that the gravitational forces between everyday objects have extremely small magnitudes. The force exerted by *m*3 on the cue ball is directed to the right:

$$\vec{F}_{31} = G\frac{m_3 m_1}{r_{31}^2}\vec{i} = (6.67\times10^{-11}\,N.m^2/kg^2)\frac{(0.3kg)(0.3kg)}{(0.3m)^2} = 6.67\times10^{-11}\vec{i}N$$

Therefore, the net gravitational force on the cue ball is

$$\vec{F} = \vec{F}_{21} + \vec{F}_{31} = (6.67\vec{i} + 3.75\vec{j})\times10^{-11}\,N$$

And the magnitude of this force is

$$\vec{F} = \sqrt{F_{21}^2 + F_{31}^2} = \sqrt{[(3.75)^2 + (6.67)^2]}\times10^{-11}\,N = 7.65\times10^{-11}\,N$$

From tan θ = *Fy/Fx* = *F21/F31* =3.75/6.67=0.562, the direction of the net gravitational force is $\theta = 29.3°$ counterclockwise from the *x* axis.

13.2 Free-Fall Acceleration and the Gravitational Force

Because the magnitude of the force acting on a freely falling object of mass *m* near the Earth's surface is given by Equation 13.4, we can equate this force *Fg* =*mg*, to obtain

$$mg = G\frac{M_E m}{R_E^2}$$

$$g = G\frac{M_E}{R_E^2} \qquad (13.5)$$

Now consider an object of mass *m* located a distance *h* above the Earth's surface or a distance *r* from the Earth's center, where *r* = *RE* + *h*. The magnitude of the gravitational force acting on this object is

$$F_g = G\frac{M_E m}{r^2} = G\frac{M_E m}{(R_E + h)^2}$$

The magnitude of the gravitational force acting on the object at this position is also *Fg* = *mg*, where *g* is the value of the free-fall acceleration at the altitude *h*. Substituting this expression for *Fg* into the last equation shows that *g* is

$$g = \frac{GM_E}{r^2} = \frac{GM_E}{(R_E + h)^2} \qquad (13.6)$$

Thus, it follows that *g decreases* with *increasing altitude.* Because the weight of an object is *mg*, we see that as $r \to \infty$, its weight approaches zero.

Example 13.2 Variation of *g* with Altitude *h*

The International Space Station operates at an altitude of 350 km. When final construction is completed, it will have a weight (measured at the Earth's surface) of 4.22×10^6 N. What is its weight when in orbit?

Solution We first find the mass of the space station from its weight at the surface of the Earth:

$$m = \frac{F_g}{g} = \frac{4.22 \times 10^6 N}{9.8 m/s^2} = 4.31 \times 10^5 kg$$

This mass is fixed—it is independent of the location of the space station. Because the station is above the surface of the Earth, however, we expect its weight in orbit to be less than its weight on the Earth. Using Equation 13.6 with h = 350 km, we obtain

$$g = \frac{GM_E}{(R_E + h)^2} = \frac{(6.67 \times 10^{-11} N.m^2/kg^2)(5.98 \times 10^{24} kg)}{(6.37 \times 10^6 m + 0.35 \times 10^6 m)^2} = 8.83 m/s^2$$

Because this value is about 90% of the value of *g* at the Earth surface, we expect that the weight of the station at an altitude of 350 km is 90% of the value at the Earth's surface.

Using the value of *g* at the location of the station, the station's weight in orbit is

$$mg = (4.31 \times 10^5 kg)(8.83 m/s^2) = 3.8 \times 10^6 N$$

Values of *g* at other altitudes are listed in Table 13.1.

13.3 Kepler's Laws and the Motion of Planets

Kepler's complete analysis of planetary motion is summarized in three statements known as Kepler's laws:

1. All planets move in elliptical orbits with the Sun at one focus.
2. The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals.
3. The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

We discuss each of these laws below.

Kepler's First Law

Kepler's first law indicates that the circular orbit is a very special case and elliptical orbits are the general situation. This was a difficult notion for scientists of the time to accept, because they felt that perfect circular orbits of the planets reflected the perfection of heaven.

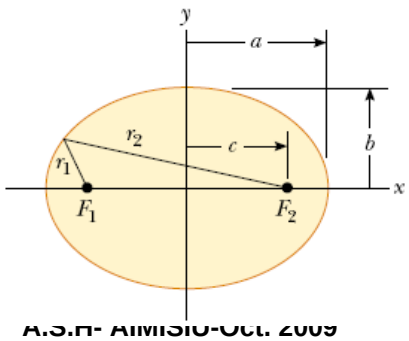


Figure 13.4 Plot of an ellipse. The semimajor axis has length *a*, and the semiminor axis has length *b*. Each focus is located at a distance *c* from the center on each side of the center.

Figure 13.4 shows the geometry of an ellipse, which serves as our model for the elliptical orbit of a planet. An ellipse is mathematically defined by choosing two points $F1$ and $F2$, each of which is a called a **focus**, and then drawing a curve through points for which the sum of the distances $r1$ and $r2$ from $F1$ and $F2$, respectively, is a constant. The longest distance through the center between points on the ellipse (and passing through both foci) is called the **major axis**, and this distance is $2a$. In Figure 13.4, the major axis is drawn along the x direction. The distance a is called the **semimajor axis**. Similarly, the shortest distance through the center between points on the ellipse is called the **minor axis** of length $2b$, where the distance b is the **semiminor axis**. Either focus of the ellipse is located at a distance c from the center of the ellipse, where $a^2 = b^2 + c^2$. In the elliptical orbit of a planet around the Sun, the Sun is at one focus of the ellipse. There is nothing at the other focus.

The **eccentricity** (اختلال مركزي) of an ellipse is defined as $e = c/a$ and describes the general shape of the ellipse. For a circle, $c = 0$, and the eccentricity is therefore zero. The smaller b is than a, the shorter the ellipse is along the y direction compared to its extent in the x direction in Figure 13.5. As b decreases, c increases, and the eccentricity e increases. Thus, higher values of eccentricity correspond to longer and thinner ellipses. The range of values of the eccentricity for an ellipse is $0 < e < 1$.

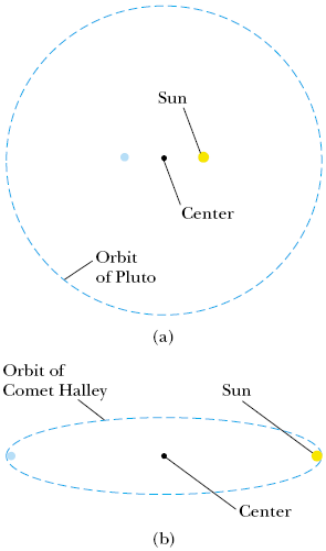
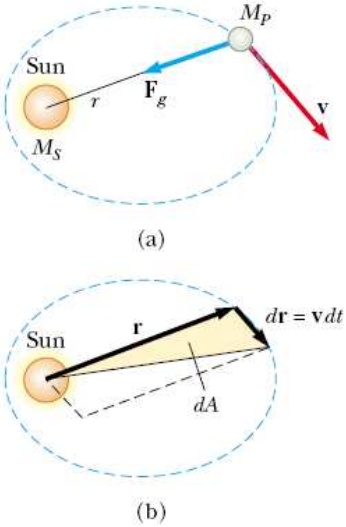


Figure 13.5 (a) The shape of the orbit of Pluto, which has the highest eccentricity ($e = 0.25$) among the planets in the solar system. The Sun is located at the large yellow dot, which is a focus of the ellipse. There is nothing physical located at the center (the small dot) or the other focus (the blue dot). (b) The shape of the orbit of Comet Halley.

Eccentricities for planetary orbits vary widely in the solar system. The eccentricity of the Earth's orbit is 0.017, which makes it nearly circular. On the other hand, the eccentricity of Pluto's orbit is 0.25, the highest of all the nine planets. Figure 13.5a shows an ellipse with the eccentricity of that of Pluto's orbit. Notice that even this highest-eccentricity orbit is difficult to distinguish from a circle. This is why Kepler's first law is an admirable accomplishment. The eccentricity of the orbit of Comet Halley is 0.97, describing an orbit whose major axis is much longer than its minor axis, as shown in Figure 13.5b. As a result, Comet Halley spends much of its 76-year period far from the Sun and invisible from the Earth. It is only visible to the naked eye during a small part of its orbit when it is near the Sun.

Kepler's Second Law

Kepler's second law can be shown to be a consequence of angular momentum conservation. Consider a planet of mass MP moving about the Sun in an elliptical orbit (Fig. 13.6a). Let us consider the planet as a system. We will model the Sun to be so much more massive than the planet that the Sun does not move. The gravitational force acting on the planet is



A.

a central force, always along the radius vector, directed toward the Sun (Fig. 13.6a).

Figure 13.6 (a) The gravitational force acting on a planet is directed toward the Sun. (b) As a planet orbits the Sun, the area swept out by the radius vector in a time interval *dt* is equal to half the area of the parallelogram formed by the vectors r and *d*r = v *dt*.

The torque on the planet due to this central force is clearly zero, because **F** is parallel to **r**. That is

$$\tau = \vec{r} \times \vec{F} = \vec{r} \times F(r)\vec{r} = 0$$

Recall that the external net torque on a system equals the time rate of change of angular momentum of the system; that is, $\tau = d\mathbf{L}/dt$. Therefore, because $\tau = 0$, **the angular momentum L of the planet is a constant of the motion:**

$$\vec{L} = \vec{r} \times \vec{p} = M_p \vec{r} \times \vec{V} = cons\tan t$$

In a time interval *dt*, the radius vector **r** in Figure 13.6b sweeps out the area *dA*, which equals half the area $|\mathbf{r}\times d\mathbf{r}|$ of the parallelogram formed by the vectors r and *d*r. Because the displacement of the planet in the time interval *dt* is given by $d\mathbf{r} = \mathbf{v}\,dt$, we have

$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times \vec{V}dt\right| = \frac{L}{2M_p}dt$$

$$\frac{dA}{dt} = \frac{L}{2M_p} = cons\tan t \qquad (13.7)$$

where *L* and *MP* are both constants. Thus, we conclude that **the radius vector from the Sun to any planet sweeps out equal areas in equal times.**

Kepler's Third Law

Consider a planet of mass *MP* that is assumed to be moving about the Sun (mass *MS*) in a circular orbit, as in Figure 13.7.

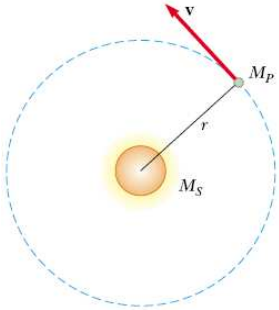


Figure 13.7 A planet of mass *MP* moving in a circular orbit around the Sun. The orbits of all planets except Mercury and Pluto are nearly circular.

Because the gravitational force provides the centripetal acceleration of the planet as it moves in a circle, we use Newton's second law for a particle in uniform circular motion $a_c = \frac{v^2}{r}$, where *ac* is the centripetal acceleration. Therefore;

$$\frac{GM_SM_P}{r^2} = M_P a_c = \frac{M_P v^2}{r} \qquad \frac{GM_S}{r^2} = \frac{v^2}{r}$$

The orbital speed of the planet is $2\pi r/T$, where *T* is the period; therefore, the preceding expression becomes

$$\frac{GM_S}{r^2} = \frac{(2\pi r/T)^2}{r}$$

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) r^3 = K_S r^3$$

where *KS* is a constant given by $K_S = \frac{4\pi^2}{GM_S} = 2.97 \times 10^{-19}\, s^2/m^3$

This equation is also valid for elliptical orbits if we replace *r* with the length *a* of the semimajor axis (Fig. 13.5):

$$T^2 = \left(\frac{4\pi^2}{GM_S}\right) a^3 = K_S a^3 \qquad \textbf{Kipler's third law} \qquad (13.8)$$

Equation 13.8 is Kepler's third law. Because the semimajor axis of a circular orbit is its radius, Equation 13.8 is valid for both circular and elliptical orbits. Note that the constant of proportionality *KS* is independent of the mass of the planet. Equation 13.8 is therefore valid for *any* planet. If we were to consider the orbit of a satellite such as the Moon about the Earth, then the constant would have a different value, with the Sun's mass replaced by the Earth's mass, that is, $KE = 4\pi^2/GME$.

Example 13.4 The Mass of the Sun

Calculate the mass of the Sun using the fact that the period of the Earth's orbit around the Sun is 3.156×10^7 s and its distance from the Sun is 1.496×10^{11} m.

Solution Using Equation 13.8, we find that

$$M_S = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2 (1.496 \times 10^{11}\, m)^3}{(6.67 \times 10^{-11}\, N.m^2/kg^2)(3.156 \times 10^7\, s)^2} = 1.99 \times 10^{30}\, kg$$

Example 13.5 A Geosynchronous Satellite (Discussed)

Consider a satellite of mass *m* moving in a circular orbit around the Earth at a constant speed *v* and at an altitude *h* above the Earth's surface, as illustrated in Figure 13.8.

(A) Determine the speed of the satellite in terms of *G*, *h*, *RE* (the radius of the Earth), and *ME* (the mass of the Earth).

(B) If the satellite is to be *geosynchronous* (that is, appearing to remain over a fixed position on the Earth), how fast is it moving through space?

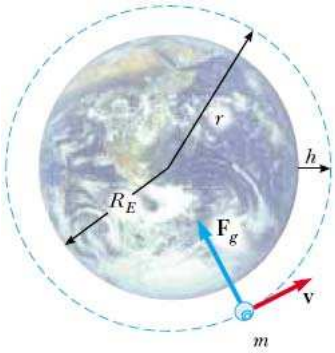


Figure 13.8 (Example 13.5) A satellite of mass *m* moving around the Earth in a circular orbit of radius *r* with constant speed *v*. The only force acting on the satellite is the gravitational force **F**g. (Not drawn to scale.)

Solution (A): Conceptualize by imagining the satellite moving around the Earth in a circular orbit under the influence of the gravitational force. The satellite must have a centripetal acceleration. Thus, we categorize this problem as one involving Newton's second law, the law of universal gravitation, and circular motion. To analyze the problem, note that the only external force acting on the satellite is the gravitational force, which acts toward the center of the Earth and keeps the satellite in its circular orbit. Therefore, the net force on the satellite is the gravitational force

$$F_g = G\frac{M_E m}{r^2}$$

From Newton's second law and the fact that the acceleration of the satellite is centripetal, we obtain

$$G\frac{M_E m}{r^2} = ma_c = m\frac{v^2}{r}$$

Solving for v and remembering that the distance r from the center of the Earth to the satellite is $r = RE + h$, we obtain

$$v = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{GM_E}{R_E + h}}$$

Solution (B): In order to appear to remain over a fixed position on the Earth, the period of the satellite must be 24 h and the satellite must be in orbit directly over the equator. From Kepler's third law (Equation 13.8) with $a = r$ and $MS \rightarrow ME$, we find the radius of the orbit:

$$T^2 = \left(\frac{4\pi^2}{GM_E}\right)r^3$$

$$r = \sqrt[3]{\frac{GM_E T^2}{4\pi^2}}$$

Substituting numerical values and noting that the period is $T = 24$ h $= 86400$ s, we find

$$r = \sqrt[3]{\frac{(6.67\times10^{-11}N.m^2/kg^2)(5.98\times10^{24}kg)(86400s)^2}{4\pi^2}} = 4.23\times10^7 m$$

To find the speed of the satellite, we use the Equation:

$$v = \sqrt{\frac{GM_E}{r}} = \sqrt{\frac{(6.67\times10^{-11}N.m^2/kg^2)(5.98\times10^{24}kg)}{4.23\times10^7 m}} = 3.07\times10^3 m/s$$

To finalize this problem, it is interesting to note that the value of r calculated here translates to a height of the satellite above the surface of the Earth of almost 36 000 km.

13.4 The Gravitational Field

An approach to describing interactions between objects that are not in contact came well after Newton's death, and it enables us to look at the gravitational interaction in a different way, using the concept of a **gravitational field** that exists at every point in space.

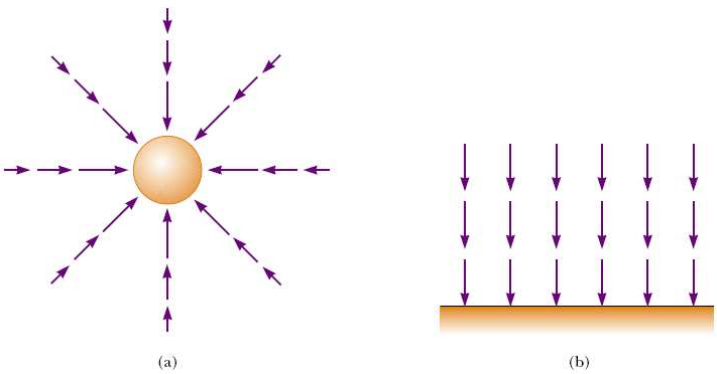


Figure 13.9 (a) The gravitational field vectors in the vicinity of a uniform spherical mass such as the Earth vary in both direction and magnitude. The vectors point in the direction of the acceleration a particle would experience if it were placed in the field. The magnitude of the field vector at any location is the magnitude of the free-fall acceleration at that location. (b) The

gravitational field vectors in a small region near the Earth's surface are uniform in both direction and magnitude.

When a particle of mass *m* is placed at a point where the gravitational field is **g**, the particle experiences a force **F***g* = *m***g**. In other words, the field exerts a force on the particle. The gravitational field **g** is defined as

$$\vec{g} \equiv \frac{\vec{F}_g}{m} \tag{13.9}$$

The Earth creates the gravitational field. We call the object creating the field the *source particle*. We can detect the presence of the field and measure its strength by placing a test particle in the field and noting the force exerted on it.

As an example of how the field concept works, consider an object of mass *m* near the Earth's surface. Because the gravitational force acting on the object has a magnitude $GMEm/r^2$ (see Eq. 13.4), the field **g** at a distance *r* from the center of the Earth is

$$\vec{g} = \frac{\vec{F}_g}{m} = -\frac{GM_E}{r^2}\vec{r} \tag{13.10}$$

where $\vec{r}$ is a unit vector pointing radially outward from the Earth and the negative sign indicates that the field points toward the center of the Earth, as illustrated in Figure 13.9a. Note that the field vectors at different points surrounding the Earth vary in both direction and magnitude. In a small region near the Earth's surface, the downward field **g** is approximately constant and uniform, as indicated in Figure 13.9b. Equation 13.10 is valid at all points *outside* the Earth's surface, assuming that the Earth is spherical. At the Earth's surface, where *r* = *RE* , **g** has a magnitude of 9.8 N/kg. (The unit N/kg is the same as m/s^2.)

13.5 Gravitational Potential Energy

The gravitational potential-energy function *mgy* for a particle–Earth system is valid only when the particle is near the Earth's surface, where the gravitational force is constant.

Recall from Equation 8.15 "$\Delta U = U_f - U_i = -\int_{x_i}^{x_f} F_x dx$" that the change in the gravitational potential energy of a system associated with a given displacement of a member of the system is defined as the negative of the work done by the gravitational force on that member during the displacement:

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F(r)dr \tag{13.11}$$

We can use this result to evaluate the gravitational potential energy function. Consider a particle of mass *m* moving between two points A and B above the Earth's surface (Fig. 13.10).

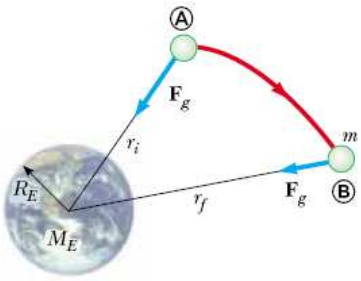


Figure 13.10 As a particle of mass *m* moves from A to B above the Earth's surface, the gravitational potential energy changes according to Equation 13.11.

The particle is subject to the gravitational force given by Equation 13.1. We can express this force as

$$F(r) = -\frac{GM_E m}{r^2}$$

where the negative sign indicates that the force is attractive. Substituting this expression for $F(r)$ into Equation 13.11, we can compute the change in the gravitational potential energy function of the particle-Earth system:

$$U_f - U_i = GM_E m \int_{r_i}^{r_f} \frac{dr}{r^2} = GM_E m \left[-\frac{1}{r} \right]_{r_i}^{r_f}$$

$$U_f - U_i = -GM_E m \left(\frac{1}{r_f} - \frac{1}{r_i} \right) \qquad (13.12)$$

It is customary to choose the reference configuration for zero potential energy to be the same as that for which the force is zero. Taking $Ui = 0$ at $ri = \infty$, we obtain the important result

$$U(r) = -\frac{GM_E m}{r} \qquad (13.13)$$

This expression applies to the Earth–particle system where the particle is separated from the center of the Earth by a distance r, provided that $r \geq RE$. The result is not valid for particles inside the Earth, where $r < RE$. Because of our choice of Ui, the function U is always negative (Fig. 13.11).

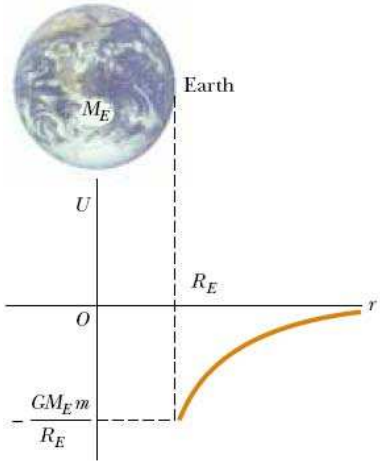


Figure 13.11 Graph of the gravitational potential energy U versus r for an object above the Earth's surface. The potential energy goes to zero as r approaches infinity.

Although Equation 13.13 was derived for the particle–Earth system, it can be applied to any two particles. That is, the gravitational potential energy associated with any pair of particles of masses $m1$ and $m2$ separated by a distance r is

$$U = -\frac{Gm_1 m_2}{r} \qquad (13.14)$$

This expression shows that the gravitational potential energy for any pair of particles varies as $1/r$, whereas the force between them varies as $1/r^2$. Furthermore, the potential energy is negative because the force is attractive and we have taken the potential energy as zero when the particle separation is infinite.

When two particles are at rest and separated by a distance r, an external agent has to supply an energy at least equal to $+Gm1m2/r$ in order to separate the particles to an infinite distance. It is therefore convenient to think of the absolute value of the potential energy as the *binding energy* of the system. If the external agent supplies an energy greater than the binding energy, the excess energy of the system will be in the form of kinetic energy of the particles when the particles are at an infinite separation.

We can extend this concept to three or more particles. In this case, the total potential energy of the system is the sum over all pairs of particles. Each pair contributes a term of the form given by Equation 13.14. For example, if the system contains three particles, as in Figure 13.12,

$$U_{total} = U_{12} + U_{13} + U_{23} = -G \left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}} \right) \qquad (13.15)$$

The absolute value of Utotal represents the work needed to separate the particles by an infinite distance.

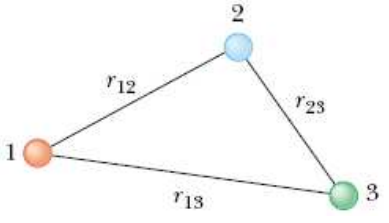


Figure 13.12 Three interacting particles.

Example 13.6 The Change in Potential Energy

A particle of mass m is displaced through a small vertical distance Δy near the Earth's surface. Show that in this situation the general expression for the change in gravitational potential energy given by Equation 13.12 reduces to the familiar relationship $\Delta U = mg\Delta y$.

Solution We can express Equation 13.12 in the form

$$\Delta U = -GM_E m\left(\frac{1}{r_f} - \frac{1}{r_i}\right) = GM_E m\left(\frac{r_f - r_i}{r_f r_i}\right)$$

If both the initial and final positions of the particle are close to the Earth's surface, then $rf - ri = \Delta y$ and $ri\ rf \approx RE^2$. (Recall that r is measured from the center of the Earth.) Therefore, the change in potential energy becomes

$$\Delta U \approx \frac{GM_E m}{R_E^2}\Delta y = mg\Delta y$$

where we have used the fact that $g = GME/RE^2$ (Eq. 13.5). Keep in mind that the reference configuration is arbitrary because it is the *change* in potential energy that is meaningful.

13.6 Energy Considerations in Planetary and Satellite Motion

Consider an object of mass m moving with a speed v in the vicinity of a massive object of mass M, where $M >> m$. The system might be a planet moving around the Sun, a satellite in orbit around the Earth, or a comet making a one-time flyby of the Sun. If we assume that the object of mass M is at rest in an inertial reference frame, then the total mechanical energy E of the two-object system when the objects are separated by a distance r is the sum of the kinetic energy of the object of mass m and the potential energy of the system, given by Equation 13.14:

$$E = K + U$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} \qquad (13.16)$$

This equation 13.16 shows that E may be positive, negative, or zero, depending on the value of v. However, for a bound system, such as the Earth–Sun system, E is necessarily *less than zero* because we have chosen the convention that $U \to 0$ as $r \to \infty$.

We can easily establish that $E<0$ for the system consisting of an object of mass m moving in a circular orbit about an object of mass $M >> m$ (Fig. 13.13).

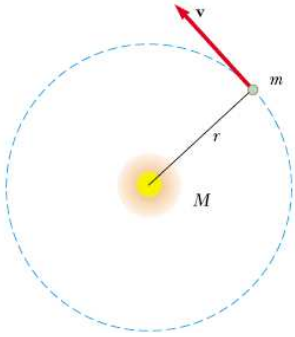


Figure 13.13 An object of mass m moving in a circular orbit about a much larger object of mass M.

Newton's second law applied to the object of mass m gives

$$F_g = \frac{GMm}{r^2} = ma = \frac{mv^2}{r}$$

Multiplying both sides by r and dividing by 2 gives

$$\frac{1}{2}mv^2 = \frac{GMm}{2r} \tag{13.17}$$

Substituting this into Equation 13.16, we obtain

$$E = \frac{GMm}{2r} - \frac{GMm}{r}$$

$$E = -\frac{GMm}{2r} \qquad \text{(circular orbits)} \tag{13.18}$$

This result clearly shows that **the total mechanical energy is negative in the case of circular orbits. Note that the kinetic energy is positive and equal to half the absolute value of the potential energy.**

The total mechanical energy is also negative in the case of elliptical orbits. The expression for E for elliptical orbits is the same as Equation 13.18 with r replaced by the semimajor axis length a:

$$E = -\frac{GMm}{2a} \qquad \text{(elliptical orbits)} \tag{13.19}$$

Furthermore, the total energy is constant if we assume that the system is isolated. Therefore, as the object of mass m moves from A to B in Figure 13.10, the total energy remains constant and Equation 13.16 gives

$$E = \frac{1}{2}mv_i^2 - \frac{GMm}{r_i} = \frac{1}{2}mv_f^2 - \frac{GMm}{r_f} \tag{13.20}$$

Combining this statement of energy conservation with our earlier discussion of conservation of angular momentum, we see that **both the total energy and the total angular momentum of a gravitationally bound, two-object system are constants of the motion.**

Example 13.7 Changing the Orbit of a Satellite

The space shuttle releases a 470 kg communications satellite while in an orbit 280 km above the surface of the Earth. A rocket engine on the satellite boosts it into a geosynchronous orbit, which is an orbit in which the satellite stays directly over a single location on the Earth. How much energy does the engine have to provide?

Solution: We first determine the initial radius (not the altitude above the Earth's surface) of the satellite's orbit when it is still in the shuttle's cargo bay. This is simply

$$RE + 280 \text{ km} = (6.37\times10^6 + 0.28\times10^6) \text{ m} = 6.65\times10^6 \text{ m} = ri$$

In Example 13.5, we found that the radius of the orbit of a geosynchronous satellite is $rf = 4.23\times10^7$ m. Applying Equation 13.18, we obtain, for the total initial and final energies,

$$E_i = -\frac{GMm}{2r_i} \qquad E_f = -\frac{GMm}{2r_f}$$

The energy required from the engine to boost the satellite is

$$\Delta E = E_f - E_i = -\frac{GM_Em}{2r_f} - \left(-\frac{GM_Em}{2r_i}\right) = -\frac{GM_Em}{2}\left(\frac{1}{r_f} - \frac{1}{r_i}\right)$$

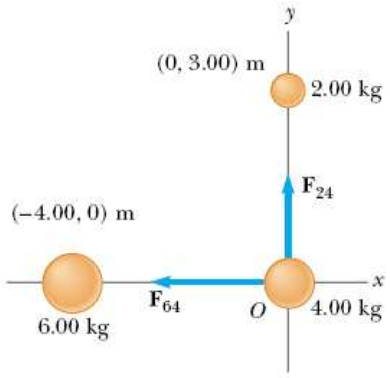
$$= -\frac{(6.67\times10^{-11}N.m^2/kg^2)(5.98\times10^{24}kg)(470kg)}{2}\times\left(\frac{1}{4.23\times10^7 m} - \frac{1}{6.65\times10^6 m}\right)$$

$= 1.19 \times 10^{10} J$

13.7 PROBLEMS

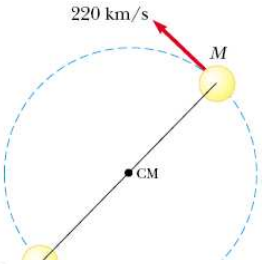
13.8.1 A 200 kg object and a 500 kg object are separated by 0.4 m. (a) Find the net gravitational force exerted by these objects on a 50 kg object placed midway between them. (b) At what position (other than an infinitely remote one) can the 50 kg object be placed so as to experience a net force of zero?

13.8.2 Three uniform spheres of mass 2 kg, 4 kg, and 6 kg are placed at the corners of a right triangle as in the opposite Figure. Calculate the resultant gravitational force on the 4 kg object, assuming the spheres are isolated from the rest of the Universe.

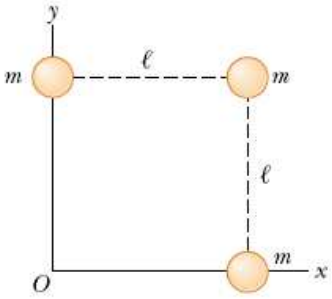


13.8.4 The free-fall acceleration on the surface of the Moon is about one sixth of that on the surface of the Earth. If the radius of the Moon is about 0.25*RE*, find the ratio of their average densities, ρMoon/ρEarth.

13.8.6 Plaskett's binary system consists of two stars that revolve in a circular orbit about a center of mass midway between them. This means that the masses of the two stars are equal (see the opposite Figure). Assume the orbital speed of each star is 220 km/s and the orbital period of each is 14.4 days. Find the mass *M* of each star.



13.8.7 Three objects of equal mass are located at three corners of a square of edge length *l* as in the opposite Figure. Find the gravitational field at the fourth corner due to these objects.



Chapter 15

Oscillatory Motion

Periodic motion is motion of an object that regularly returns to a given position after a fixed time interval.
A special kind of periodic motion occurs in mechanical systems when the force acting on an object is proportional to the position of the object relative to some equilibrium position. If this force is always directed toward the equilibrium position, the motion is called *simple harmonic motion*, which is primary focus of this chapter.

15.1 Motion of an Object Attached to a Spring

As a model for simple harmonic motion, consider a block of mass *m* attached to the end of a spring, with the block free to move on horizontal, frictionless surface (Figure 15.1). When the spring is neither stretched nor compressed, the block is at the position called the **equilibrium position** of the system, which we identify $x = 0$. We know from experience that such system oscillates back and forth if disturbed from its equilibrium position.

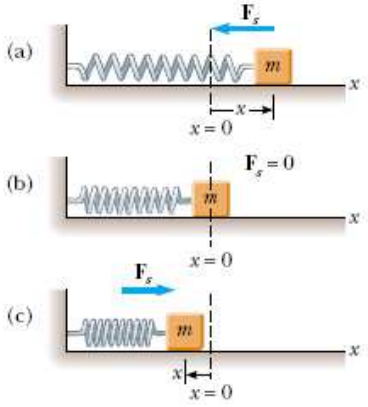


Figure 15.1 A block attached to a spring moving on a frictionless surface. (a) When the block is displaced to the right of equilibrium ($x > 0$), the force exerted by the spring act to the right. (b) When the block is at its equilibrium position ($x = 0$), the force exerted by the spring is zero. (c) When the block is displaced to the left of equilibrium ($x < 0$), the force exerted by the spring acts to the right.

We know from section 7.4 that the motion of Figure 15.1 is given by **Hook's law**:

$$F_S = - kx \qquad (15.1)$$

We call this a **restoring force** because is always directed toward the equilibrium position and therefore *opposite* the displacement from the equilibrium.
Applying Newton's second law $\sum F_x = ma_x$ to the motion of the block, with Equation 15.1, we obtain

$$-kx = ma_x$$

$$a_x = -\frac{k}{m}x \qquad (15.2)$$

The system is behaves in **simple harmonic motion.** If the block in Figure 15.1 is displaced to a position $x = A$ and released from rest, its *initial* acceleration is $-kA/m$. When the block passes through the equilibrium position $x = 0$, its acceleration is zero. At this instant, its speed is a maximum. The block continuous travel to the left of equilibrium and finally reaches $x = - A$, at which its acceleration is $+kA/m$ and its speed is again zero, as it is discussed in section 7.4 and 8.6. The block complete a full cycle of its motion by returning to the original position, again passing trough $x = 0$

with maximum speed. Therefore, the block oscillates between the turning points $x = \pm A$. In the absence of friction, because the force exerted by the spring is conservative, this idealized motion will continue forever.

15.2 The Particle in Simple Harmonic Motion

We model the block as a particle subject to the force in Equation 15.1. By definition,

$$a = dv/dt = d^2x/dt^2$$

We can express Equation 15.2 as

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{15.3}$$

If we choose the ratio k/m with the symbol ω^2 (we choose ω^2 rather than ω in order to make the solution that we develop below simpler in form), then

$$\omega^2 = \frac{k}{m} \tag{15.4}$$

and Equation 15.3 can be written in the form

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{15.5}$$

We seek a function $x(t)$ whose second derivative is the same as the original function with negative sign and multiplied by ω^2. The following cosine function is a solution to the differential equation:

$$x(t) = A\cos(\omega t + \phi) \tag{15.6}$$

Position versus time for an object in simple harmonic motion

Where A, ω, and ϕ are constants. To see explicitly this equation satisfies Equation 15.5, note that

$$\frac{dx}{dt} = A\frac{d}{dt}\cos(\omega t + \phi) = -\omega A\sin(\omega t + \phi) \tag{15.7}$$

$$\frac{d^2x}{dt^2} = -A\frac{d}{dt}\sin(\omega t + \phi) = -\omega^2 A\cos(\omega t + \phi) \tag{15.8}$$

Comparing Equations 15.6 and 15.8, we see that $d^2x/dt^2 = -\omega^2 x$ and Equation 15.5 is satisfied.

In order to give a physical significance for the constants A, ω, and ϕ, it is convenient to form a graphical representation of the motion by plotting x as a function of t, as in Figure 15.2a.

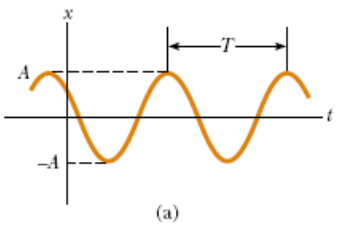
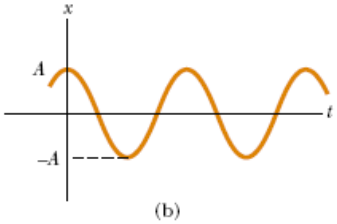


Figure 15.2 (a) An x-vs-t graph for an object undergoing simple harmonic motion. The amplitude of the motion is A, the period is T, and the phase constant is Ø. (b) The x-vs-t graph in special case in which $x = A$ at $t = 0$ and hence Ø= 0.



First, note that A, called the **amplitude** of the motion, is simply **the maximum value of the position of the particle in either positive or negative x direction.** The constant ω

is called the **angular frequency,** and has unit rad/s. The more oscillations per unit time, the higher is the value of ω. From equation 15.4, the angular frequency is

$$\omega = \sqrt{\frac{k}{m}} \tag{15.9}$$

The constant angle ϕ is called the **phase constant** and along with amplitude A, is determine uniquely by the position of velocity of the particle at $t = 0$. If the particle at its maximum position $x = A$ at $t = 0$, the phase constant is $\phi = 0$ and the graphical representation of the motion is shown in Figure 15.2b. The quantity $(\omega t + \phi)$ is called the **phase** of the motion.
Equations 15.1, 15.5, and 15.6 form the basis of the mathematical representation of the particle in simple harmonic motion model.

The **period** T of the motion is time interval required for the particle to go through one full cycle of its motion (Fig. 15.2a). That is, the values of x and v for the particle at time t equal the values of x and v at time $t + T$. Because the phase increases by 2π radians in a time interval of T:

$$[\omega(t+T)+\phi]-(\omega t+\phi) = 2\pi$$

Simplifying this expression, we see that $\omega T = 2\pi$, or

$$T = \frac{2\pi}{\omega} \tag{15.10}$$

The inverse of the period is called the **frequency** f of the motion. Whereas the period is the time interval per oscillation, the frequency represents the **number of oscillations that the particle undergoes per unit time interval:**

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{15.11}$$

The units of f are cycles per second, or **hertz** (Hz). Rearranging Equation 15.11 gives

$$\omega = 2\pi f = \frac{2\pi}{T} \tag{15.12}$$

Equations 15.9, 15.10, and 15.11 can be used to express the period and frequency of the motion for the particle in simple harmonic motion in terms of the characteristics m and k of the system as

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{Period} \tag{15.13}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \text{Frequency} \tag{15.14}$$

That is, the period and frequency depend *only* on the mass of the particle and the force constant of the spring. The frequency is larger for a stiffer spring (larger value of k) and decreases with increasing mass of the particle.
We can obtain the velocity and acceleration of a particle undergoing simple harmonic motion from Equation 15.7 and 15.8:

$$v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi) \quad \text{Velocity of an object in simple harmonic motion} \tag{15.15}$$

$$a = \frac{d^2x}{dt^2} = -\omega^2 A\cos(\omega t+\phi) \quad \text{Acceleration of an object in simple harmonic motion} \tag{15.16}$$

From Equation 15.5 we see that, because the sine and cosine functions oscillate between the ±1, extreme values of the velocity v are $\pm\omega A$. Likewise, Equation 15.16 shows that the extreme values of the acceleration a are $\pm\omega^2 A$. Therefore the *maximum* values of the magnitudes of the velocity and acceleration are

$$v_{max} = \omega A = \sqrt{\frac{k}{m}}A \qquad (15.17)$$

Maximum magnitudes of speed and acceleration in simple harmonic motion

$$a_{max} = \omega^2 A = \frac{k}{m}A \qquad (15.18)$$

Figure15.4a plots position versus time for an arbitrary value of the phase constant. The associated velocity-time and acceleration-time curves are illustrated in Figures 15.4b and 15.4c. They show that the phase of velocity differs from the phase of the position by π/2 rad, or 90°. That is, when x is a maximum or a minimum, the velocity is zero. Likewise, when x is zero, the speed is a maximum. Furthermore, notice that the phase of acceleration differs from phase of the position by π radians, or 180°. For example, when x is a maximum, a has a maximum magnitude in the opposite direction.

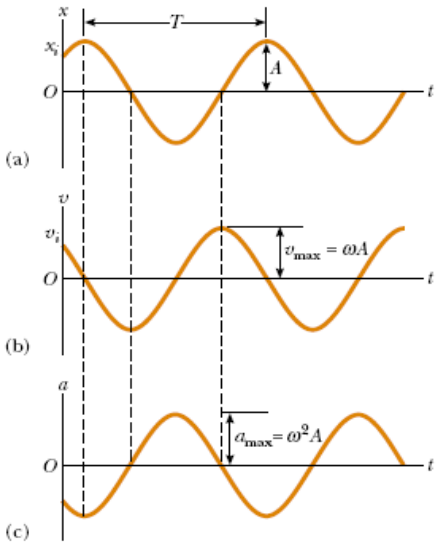


Figure 15.4 Graphical representation of simple harmonic motion. (a) Position versus time. (b) Velocity versus time. (c) Acceleration versus time. Note that at any specified time the velocity is 90° out of phase with the position and acceleration is 180° out of the phase with the position.

Suppose the particle is set into a motion by pulling the particle from equilibrium by a distance A and releasing it from rest at t = 0, as in Figure 15.5.

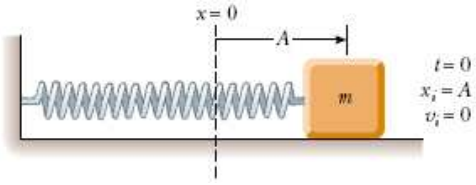


Figure 15.5 A block–spring system that begins its motion from rest with the block at $x = A$ at t = 0. In this case $\phi = 0$ and $x = A\cos\omega t$.

We must then require our solution for $x(t)$ and $v(t)$ (Eqs. 15.6 and 15.15) obey the initial conditions that $x(0) = A$ and $v(0) = 0$:

$$x(0) = A\cos\phi = A$$

$$v(0) = -\omega A\sin\phi = 0$$

These conditions are met if we choose $\phi = 0$, giving $x = A\cos\omega t$. To check this solution, note that it satisfies the condition that $x(0)$ = A, because cos 0 = 1.

From Figure 15.6a, the acceleration reaches extreme values $\pm\omega^2 A$ when the position has extreme values $\pm A$. Furthermore, the velocity has the extreme values of $\pm\omega A$, which both occur at $x = 0$.

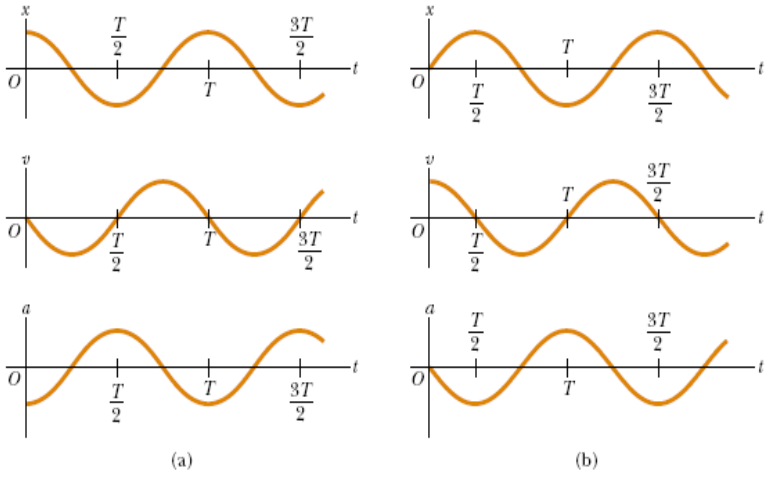


Figure 15.6 (a) position, velocity, and acceleration versus time for a block undergoing simple harmonic motion under the initial conditions that at $t = 0$, $x(0) = A$ and $v(0) = 0$. (b) Position, velocity, and acceleration versus time for a block undergoing simple harmonic motion under the initial conditions that at $t = 0$, $x(0) = 0$, and $v(0) = v_i$.

Suppose that the system is oscillating and we define $t = 0$ as the instant that the particle passes through the unstretched position of the spring while moving to the right (Figure 15.7).

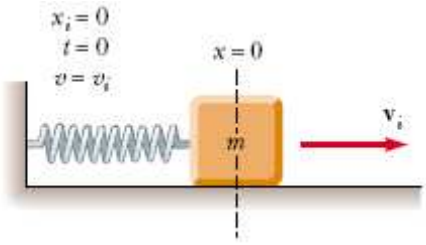


Figure 15.7 The block-spring system is undergoing oscillation, and $t = 0$ is defined at an instant when the block passes through the equilibrium position $x = 0$ and is moving to the right with speed v_i.

In this case we must require that our solutions for $x(t)$ and $v(t)$ obey the initial conditions that $x(0) = 0$ and $v(0) = v_i$:

$$x(0) = A\cos\phi = 0$$
$$v(0) = -\omega A\sin\phi = v_i$$

The first of these conditions tells us that $\phi = \pm\pi/2$ (if cos ø = 0, then ø = π/2 or 90°). With these choices for ø, the second condition tells us that $A = \pm v_i/\omega$ (from the 1st condition ø = π/2, therefore sin ø =1). Because the initial velocity is positive and the amplitude must be positive, we must have ø = - π/2. Hence, the solution is given by

$$x = \frac{v_i}{\omega}\cos(\omega t - \frac{\pi}{2})$$

The graph of position, velocity, and acceleration versus time for this choice of $t = 0$ are shown in Figure 15.6b. Note that these curves are the same as those in Figure 15.6a, but shifted to the right by one fourth of the cycle.

15.3 Energy of the simple Harmonic Oscillator

Let us examine the mechanical energy of the block-spring system illustrated in Figure 15.1. Because the surface is frictionless, we expect the total mechanical energy of the

system to be constant. We assume a massless spring, so the kinetic energy of the system corresponds only to that of the block. We can use Equation 15.15 to express the kinetic energy of the block as

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$ Kinetic energy of a simple harmonic oscillator (15.19)

The elastic potential energy stored in the spring for any elongation x is given by $\frac{1}{2}kx^2$. Using Equation 15.6, we obtain

$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$ Potential energy of a simple harmonic oscillator (15.20)

We see that K and U are *always* positive quantities. Because $\omega^2 = k/m$, we can express total mechanical energy of the simple harmonic oscillator as

$$E = K + U = \frac{1}{2}kA^2[\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)]$$

From the identity $\sin^2\theta + \cos^2\theta = 1$. Therefore, this equation reduced to

$$E = \frac{1}{2}kA^2$$ Total energy of a simple harmonic oscillator (15.21)

That is, **the total mechanical energy of a simple harmonic oscillator is a constant of the motion and is proportional to the square of the amplitude.** U is small when K is large, and vise versa, because the sum must be constant. U is maximum when $x = \pm A$ because $v = 0$ at these points and thus there is no K. At equilibrium position, where $U = 0$ because $x = 0$, the E, all in the form of K, is again $\frac{1}{2}kA^2$. That is,

$$E = \frac{1}{2}mv_{\max}^2 = \frac{1}{2}m\omega^2 A^2 = \frac{1}{2}m\frac{k}{m}A^2 = \frac{1}{2}kA^2 \qquad (\text{at } x = 0)$$

Plots of the kinetic and potential energies versus time appear in Figure 15.8a, after we have taken ø = 0. The variation of K and U with position x of the block are plotted in Figure 15.8b.

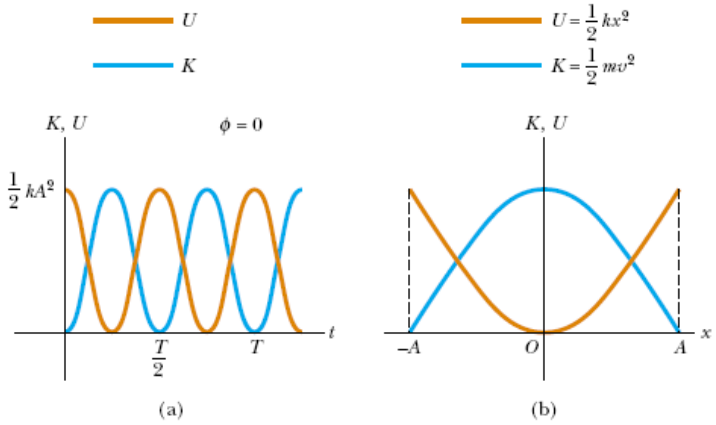


Figure 15.8 (a) kinetic energy and potential energy versus time for a simple harmonic oscillator with $\phi = 0$. (b) Kinetic energy and potential energy versus position for a simple harmonic oscillator. In either plot, note that $K + U =$ constant.

Figure 15.9 illustrates the position, velocity, acceleration, kinetic energy, and potential energy of the block-spring system for one full period of the motion.

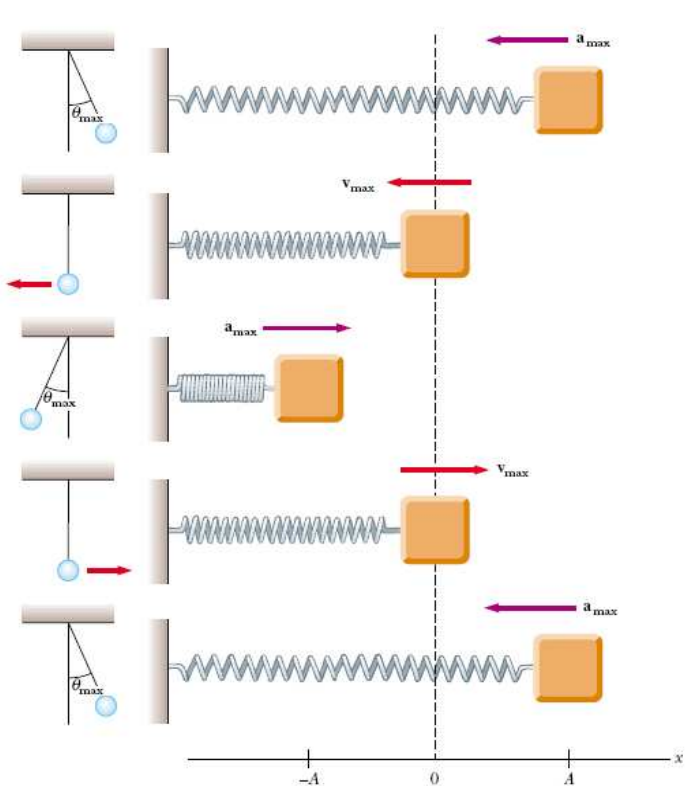
Finally, we can use the principle of conservation of energy to obtain the velocity for an arbitrary position by expressing the total energy at some arbitrary position x as

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2$$

$$v = \pm\sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\omega\sqrt{A^2 - x^2} \qquad (15.22)$$

Velocity as a function of position for simple harmonic oscillator

From Equation 15.22, we find that the speed is a maximum at $x = 0$ and is zero at the turning points $x = \pm A$. Simple harmonic oscillator are good models of a wide variety of physical phenomena.



t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Figure 15.9 Simple harmonic motion for a block-spring system and its analogy to the motion of a simple pendulum. The parameters in the table at the right refer to the block-spring system, assuming that at $t = 0$, $x = A$ so that $x = A\cos\omega t$.

15.4 Comparing Simple Harmonic Motion with Uniform Circular Motion

Some common devices in our everyday life exhibit a relationship between oscillatory motion and circular motion. In Figure 15.10 ball is attached to the rim of a turntable of radius A, which is illuminated from the side by a lamp. The ball casts a shadow on a screen. **As the turntable rotates with constant angular speed, the shadow of the ball moves back and forth in simple harmonic motion.**

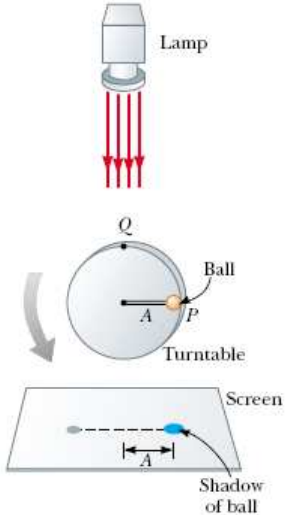


Figure 15.10 experimental set up for demonstrating the connection between simple harmonic motion and uniform circular motion. As the ball rotates on the turntable with constant angular speed, its shadow on the screen move back and forth in simple harmonic motion.

Consider a particle located at point P on the circumference of a circle of radius A, as in Figure 15.10a, with the line OP making an angle ϕ with the x axis at $t = 0$. If the particle moves along the circle with constant speed ω until OP makes an angle θ with the x axis, as in Figure 15.11b, then at some time $t > 0$, the angle between OP and the x axis is $\theta = \omega t + \phi$. As the particle moves along the circle, the projection of P on the x axis, labeled point Q, moves back and forth along the x axis between the limits $x = \pm A$. Note that points P and Q always have the same x coordinate. From the right triangle OPQ, we see that this x coordinate is

$$x(t) = A\cos(\omega t + \phi) \tag{15.23}$$

This expression is the same as Equation 15.6 and show that the point Q moves with simple harmonic motion along the x axis. Therefore, **simple harmonic motion along a straight line can be represented by the projection of a uniform circular motion along diameter of a reference circle.**

This geometrical interpretation shows that the time interval for one complete revolution of the point P on the reference circle is equal to the period of motion T for simple harmonic motion between $x = \pm A$.

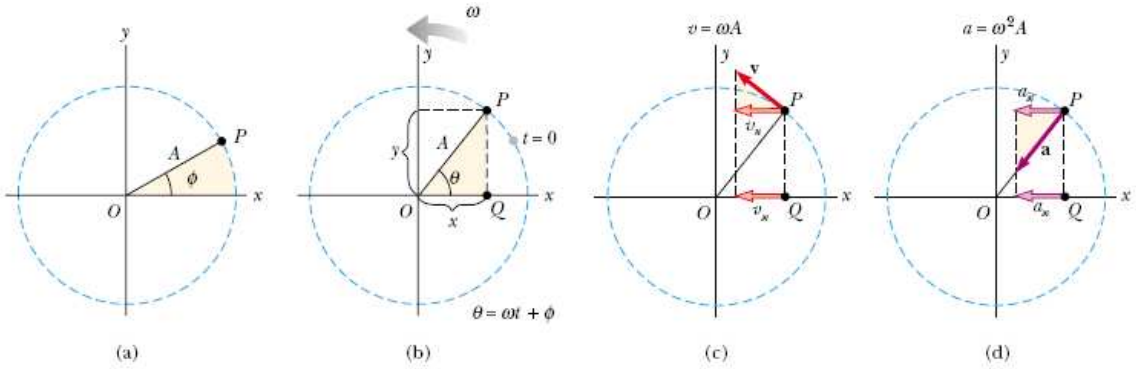


Figure 15.11 Relationship between the uniform circular motion of a point P and the simple harmonic motion f a point Q.

15.5 The Pendulum

The **simple pendulum** is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end, as shown in Figure 15.12.

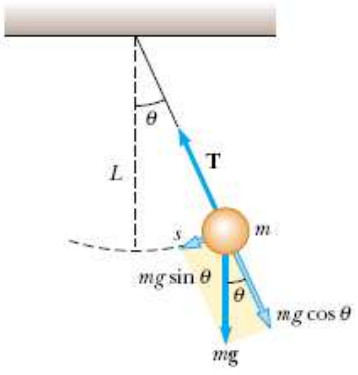


Figure 15.12 When θ is small, a simple pendulum oscillates in simple harmonic motion about the equilibrium position $\theta = 0$. The restoring force is $-mg \sin\theta$, the component of the gravitational force tangent to the arc.

The forces acting on the bob are the force **T** exerted by the string and the gravitational force mg. The tangential component $mg \sin\theta$ of the gravitational force always acts toward $\theta = 0$. We can apply Newton's second law for motion in the tangential direction:

$$F_t = -mg\sin\theta = m\frac{d^2s}{dt^2}$$

Where s is the bob's position measured along the arc and the negative sign indicates that the toward the equilibrium (vertical) position. Because $s = L\theta$ (Eq. 10.1a, for rotating disc: $s = r\theta$) and L is constant, after replacing s by θ, this equation reduces to

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

If we assume θ is *small* (less than 10^o or 0.2 rad), however, we can use the **small angle approximation**, in which $\sin\theta \approx \theta$.
Therefore the equation of the motion for the simple pendulum becomes

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta \qquad \text{(for small values of } \theta\text{)} \qquad (15.24)$$

Equation 15.24 has the same form as Equation 15.3, so we conclude that the motion for small amplitudes of oscillation can be modeled as simple harmonic motion.
By replacing k and m in Equation 15.4 by g and L, we get

$$\omega = \sqrt{\frac{g}{L}} \qquad (15.25)$$

The period of the motion is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}} \qquad (15.26)$$

In other words, **the period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity.** Because the period is independent of the mass, we conclude that all simple pendulua that are of equal length and are at the same location (so that g is constant) oscillate with same period. The simple pendulum can be used as a time keeper.

Physical Pendulum

If a hanging object oscillates about a fixed axis that dose not pass through its centre of mass and the object cannot be approximated as a point mass and the object cannot be approximated as a point mass, we cannot treat the system as a simple pendulum. In this case, the system is called a **physical pendulum.**

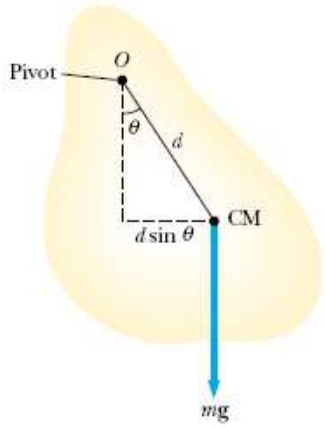


Figure 15.13 A physical pendulum pivoted at O.

Consider a rigid object pivoted at a point O that is a distance d from the centre of mass (Figure 15.13). The gravitational force provides a torque about an axis through O, and the magnitude of the torque is

$$mgd\sin\theta$$

Using the rotational form of Newton's second law, $\sum\tau = I\alpha$, where I is the moment of inertia about the axis through O, we obtain

$$-mgd\sin\theta = I\frac{d^2\theta}{dt^2}$$

The negative sign indicates that the torque about O tends to decrease θ. If we again assume that θ is small, then $\sin\theta \approx \theta$ is valid, and the equation of motion reduces to

$$\frac{d^2\theta}{dt^2} = -(\frac{mgd}{I})\theta = -\omega^2\theta \qquad (15.27)$$

Because this equation of the same form as Equation 15.3, the motion is simple harmonic motion, therefore

$$\omega = \sqrt{\frac{mgd}{I}}$$

The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgd}} \qquad \text{Period of a physical pendulum} \qquad (15.28)$$

The result can be used to measure the moment of inertia of a flat rigid object. If the location of the centre of mass—and hence the value of d—is known, the moment of inertia can be obtained by measuring the period.

Torsional Pendulum

Figure 15.14 shows a rigid object suspended by a wire attached at the top of a fixed support.

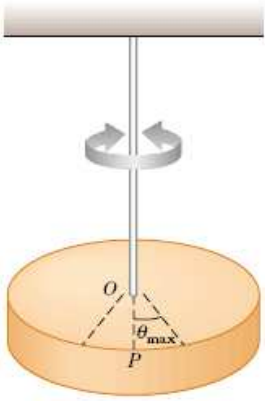


Figure 15.14 A torsional pendulum consists of a rigid object suspended by a wire attached to a rigid support. The object oscillates about the line OP with an amplitude θ_{max}.

When the object twisted through some angle θ, the twisted wire exerts on the object a restoring torque that is proportional to the angular position. That is,

$$\tau = -\kappa\theta$$

Where κ (kappa) is called the *torsion constant* of the support wire. Applying Newton's second law for rotational motion (which is $\tau = -mgd\sin\theta = I\frac{d^2\theta}{dt^2}$), we find

$$\tau = -\kappa\theta = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{\kappa}{I}\theta \qquad (15.29)$$

Again, this is the equation of motion for a simple harmonic oscillator, with $\omega = \sqrt{\kappa / I}$ and a period

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \qquad \text{Period of a torsional pendulum} \qquad (15.30)$$

This system is called *torsional pendulum*. There is no small-angle restriction in this situation as long as the elastic limit of the wire not exceeded.

15.7 Problems

P15.2/4 In an engine, a piston oscillates with simple harmonic motion so that its position varies according to the expression: x = 5cm cos (2t + π/6), where x is in centimeters and t in seconds. At t = 0, find (a) the position of the piston, (b) its velocity, and (c) its acceleration. (d) Find the period and amplitude of the motion.

P15.3 The position of a particle is given by the expression: x=(4m)cos(3πt +π), where x is in meters and t is in seconds. Determine (a) the frequency and the period of the motion, (b) the amplitude of the motion, (c) the phase constant, and (d) the position the particle at t = 0.25 s.

$$x = (4cm)\cos(1.75\pi) = 4m\cos(1.75\times 180) = 4m\cos(315) = 4m\times 0.707 = 2.83m$$

P15.16/12 A 200 g block is attached to a horizontal spring and executes simple harmonic motion with a period of 0.25 s. If the total energy of the system is 2 J, find (a) the force constant of the spring and (b) the amplitude of the motion.

P15.25/21 While riding behind a car traveling at 3 m/s, you notice that one of the car's tires has a small hemispherical bump on its rim, as in Figure P15.25. (a) Explain why the bump, from your viewpoint behind the car, executes simple harmonic motion. (b) If the radii of the car's tires are 0.3 m, what is the bump's period of oscillation?

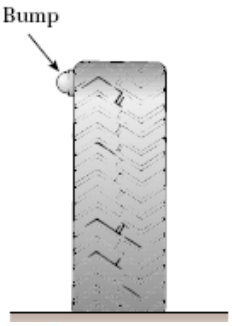


Figure P15.25

P15.35/25 A physical pendulum in the form of a planar body moves in simple harmonic motion with a frequency of 0.45 Hz. If the pendulum has a mass of 2.2 kg and the pivot is located 0.35 m from the center of mass, determine the moment of inertia of the pendulum about the pivot point.